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Philosophy of Mathematics –
Set Theory, Measuring Theories, and Nominalism

WWW.PROTOSOCIOLOGY.DE

Frankfurt am Main
http://www.protosociology.de
peter@protosociology.de

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Who's Afraid of Inconsistent Mathematics?

Mark Colyvan

Abstract
Contemporary mathematical theories are generally thought to be consistent. But it hasn't always been this way; there have been times in the history of mathematics when the consistency of various mathematical theories has been called into question. And some theories, such as naïve set theory and (arguably) the early calculus, were shown to be inconsistent. In this paper I will consider some of the philosophical issues arising from inconsistent mathematical theories.

1. A Five Line Proof of Fermat's Last Theorem

Fermat's Last Theorem says that there are no positive integers x, y, and z, and integer $n > 2$, such that $x^n + y^n = z^n$. This theorem has a long and illustrious history but was finally proven in the 1990s by Andrew Wiles. Despite the apparent simplicity of the theorem itself, the proof runs over a hundred pages, invokes some very advanced mathematics (the theory elliptic curves, amongst other things), and is understandable to only a handful of mathematicians.[1] But now consider the following proof.

Fermat's Last Theorem (FLT): There are no positive integers x, y, and z, and integer $n > 2$, such that $x^n + y^n = z^n$.

Proof: Let $\mathcal{R}$ stand for the Russell set, the set of all sets that are not members of themselves: $\mathcal{R} = \{x : x \notin x\}$. It is straightforward to show that this set is both a member of itself and not a member of itself: $\mathcal{R} \in \mathcal{R}$ and $\mathcal{R} \notin \mathcal{R}$. Since $\mathcal{R} \in \mathcal{R}$, it follows that $\mathcal{R} \in \mathcal{R}$ or FLT. But since $\mathcal{R} \notin \mathcal{R}$, by disjunctive syllogism, FLT.

This proof is short, easily understood by anyone with just a bit of high-school mathematics. Moreover, the proof was available to mathematicians well before Wiles' groundbreaking research. Why wasn't the above proof ever advanced? One reason is that the proof invokes an inconsistent mathematical theory, namely, naïve set theory. This theory was shown to be inconsistent toward the end of the 19th century. The most famous inconsistency arising in it was a para-

1 See S. Singh, *Fermat's Last Theorem: The Story of a Riddle that Confounded the World's Greatest Minds for 358 Years,* London 1997, for a popular account of Fermat's Last Theorem.

dox due to Bertrand Russell. I invoked Russell's paradoxical set in the above proof.[2] Paradoxes such as Russell's (and, to a lesser extent, others such as the Burali-Forti ordinal paradox and Cantor's cardinality paradox) led to a crisis in mathematics at the turn of the 20th Century. This, in turn, led to many years of sustained work on the foundations of mathematics. In particular, a huge effort was put into finding a consistent (or at least not known-to-be-inconsistent) replacement for naïve set theory. The generally-agreed-upon replacement is Zermelo-Fraenkel set theory with the axiom of choice (ZFC).[3]

But the inconsistency of naïve set theory cannot be the whole story of why the above proof of Fermat's Last Theorem was never seriously advanced. After all, there was a period of some 30 odd years between the discovery of Russell's paradox and the development of ZFC. Mathematicians did not shut up shop until the foundational questions were settled. They continued working, using naïve set theory, albeit rather cautiously. Moreover, it might be argued that many mathematicians to this day, still use naïve set theory.[4] In summary, we have a situation where mathematicians knew about the paradoxes and they continued to use a known-to-be-inconsistent mathematical theory in the development of other branches of mathematics and in applications beyond mathematics.

This raises a number of interesting philosophical questions about inconsistent mathematics, its logic and its applications. I'll pursue two of these issues in this paper. The first concerns the logic used in mathematics. It is part of the accepted wisdom that in mathematics, classical logic is king. Despite a serious challenge from the intuitionists in the early part of the twentieth century, classical logic is generally thought to have prevailed. But now we have a new challenge from logics more tolerant to inconsistency, so-called *paraconsistent logics*. In the next section I will give a brief outline of paraconsistent logics and discuss their relevance for the question of the appropriate logic for mathemat-

2 The paradox is that the Russell set both is and is not a member of itself.

3 See M. Giaquinto, *The Search for Certainty: A Philosophical Account of Foundations of Mathematics,* Oxford 2002, for an account of the history and H. B. Enderton, *Elements of Set Theory,* New York 1997, for details of ZFC set theory.

4 After all, so long as you are careful to skirt around the known paradoxes of naïve set theory, it can be safely used in areas such as analysis, topology, algebra and the like. Most mathematical proofs, outside of set theory, do not explicitly state the set theory being employed. Moreover, typically these proofs do not show how the various set-theoretic constructions are legitimate according to ZFC. This suggests, at least, that the background set theory is naïve, where there are less restrictions on set-theoretic constructions. See Enderton, 1997 and P. R. Halmos, *Naïve Set Theory*, New York 1974, for the details.

ics. I will suggest that not only are such logics appropriate, but they may already be the logic of choice amongst the mathematical community.

The second general topic I will discuss concerns applications of inconsistent mathematics, both within mathematics itself and in empirical science. There are many questions here but I will focus on two: how can an inconsistent theory apply to a presumably consistent world?; and what do the applications of inconsistent mathematical theories tell us about what exists? But before we broach such philosophical matters, I will first present a couple of examples of inconsistent mathematical theories.

2. Inconsistent Mathematics

We have already seen Russell's paradox, the paradox arising from the set of all sets that are not members of themselves: $\mathcal{R} = \{x : x \notin x\}$. The paradox arises because of an axiom of naïve set theory known as *unrestricted comprehension*. This axiom says that for every predicate, there is a corresponding set. So, for example, there is the predicate "is a cat" and there is the set of all cats; there is the predicate is a "natural number" and there is the set of all natural numbers. So far, so good. The trouble starts when we consider predicates such as "is a set" or "is a non-self-membered set". If there are sets corresponding to these two predicates, we get Cantor's cardinality paradox and Russell's paradox, respectively. Cantor's cardinality paradox starts by assuming that there is a set of all sets, Ω, with cardinality[5] ω. Now consider the power set of Ω: $\wp(\Omega)$. Cantor's theorem can be invoked to show that the cardinality of $\wp(\Omega)$ is strictly greater than the cardinality of Ω. But Ω is the set of all sets and so must have cardinality at least as large as any set of sets. Since $\wp(\Omega)$ is a set of sets, we have a contradiction.

The naïve axiom of unrestricted comprehension was seen to be the culprit in all the paradoxes, and mathematicians set about finding ways to limit the scope of this overly powerful principle. One obvious suggestion is to simply ban the problematic sets—like the set of all sets and Russell's set. This, however, is clearly *ad hoc*. Slightly better is to ban all sets that refer to themselves (either explicitly or implicitly) in their own specification. The generally-agreed-upon solution achieves the latter by invoking axioms that insure that such problem-

5 This, in a mathematically precise sense, is the "size" of the set.

atic sets (and others as well) cannot be formed. This is ZFC. The basic idea is to have a hierarchy of sets, where sets can only be formed from sets of a lower level—a set cannot have itself as a member, for instance, because that would involve collecting sets from the same level. Nor can there be a set of all sets—only a set of all sets from lower down in the hierarchy. ZFC has not engendered any paradoxes but it has the look and feel of a theory designed to avoid disaster rather than a natural successor to naïve set theory. More on this later.

Another important example of an inconsistent mathematical theory is the early calculus. When the calculus was first developed in the late 17th century by Newton and Leibniz, it was fairly straightforwardly inconsistent. It invoked strange mathematical items called infinitesimals (or fluxions). These items are supposed to be changing mathematical entities that approach zero. The problem is that in some places these entities behave like real numbers close to zero but in other places they behave like zero. Take an example from the early calculus: differentiating a polynomial such as $f(x) = ax^2 + bx + c$.[6]

$$f'(x) = \frac{f(x+\delta) - f(x)}{\delta} \tag{1}$$

$$= \frac{a(x+\delta)^2 + b(x+\delta) + c - (ax^2 + bx + c)}{\delta} \tag{2}$$

$$= \frac{2ax\delta + \delta^2 + b\delta}{\delta} \tag{3}$$

$$= 2ax + b + \delta \tag{4}$$

$$= 2ax + b \tag{5}$$

Here we see that at lines 1–3 the infinitesimal δ is treated as non-zero, for otherwise we could not divide by it. But just one line later we find that $2ax + b + \delta = 2ax + b$, which implies that $\delta = 0$. The dual nature of such infinitesimals can lead to trouble, at least if care is not exercised. After all, if infinitesimals behave like zero in situation like lines 4 and 5 above, why not allow:

$$2 \times \delta = 3 \times \delta$$

6 The omission of the limit $\lim_{\delta \to 0}$ from the right-hand side on the first four lines of the following calculation is deliberate. Such limits are a modern development. At the time of Newton and Leibniz, there was no rigorous theory of limits; differentiating from first principles was along the lines presented here.

then divide by δ to yield

$2 = 3$?

This illustrates how easily trouble can arise and spread if 17th and 18th century mathematicians weren't careful. There were rules about how these inconsistent mathematical objects, infinitesimals, were to be used. And according to the rules in question, the first calculation above is legitimate but the second is not. No surprises there. But one can quite reasonably ask after the motivation for the rules in question. Such rules about what is legitimate and what is not require motivation beyond what does and what does not lead to trouble.

The calculus was eventually, and gradually, made rigorous by the work of Bolzano, Cauchy, Weierstrass, and others[7] in the 19th century. This was achieved by a rigorous (ε–δ) definition of limit.[8] So, to be clear, I am not claiming that there are any ongoing consistency problems for the calculus. The point is simply that for over a hundred years mathematicians and physicists worked with what would seem to be an inconsistent theory of calculus.[9]

3. Is the Appropriate Logic for Mathematics Paraconsistent?

Classical logic has it that an argument form known as *ex contradictione quodlibet* or *explosion* is valid. The argument form was used in my proof of Fermat's Last Theorem at the beginning of this paper. According to explosion any arbitrary proposition follows from a contradiction.[10] Logics in which this argument form is valid are said to be *explosive*. A *paraconsistent logic* is one that is not

7 M. Kline, *Mathematical Thought from Ancient to Modern Times,* New York 1972.

8 More recently there has been a revival of something like the original infinitesimal idea by A. Robinson, *Non-standard Analysis,* Amsterdam 1966, and J. H. Conway, *On Numbers and Games,* New York 1976, and even an explicitly inconsistent theory of infinitesimals by C. Mortensen, *Inconsistent Mathematics,* Dortrecht 1995.

9 There are also cases where explicitly inconsistent, but non-trivial, theories have been developed. See R. K. Meyer, "Relevant Arithmetic", *Bulletin of the Section of Logic of the Polish Academy of Sciences* 1976, 5:133–137; R. K. Meyer, and C. Mortensen, "Inconsistent Models for Relevant Arithmetic", *Journal of Symbolic Logic* 1984, 49: 917–929; C. Mortensen, 1995; G. Priest, "Inconsistent Models of Arithmetic Part I: Finite Models", *Journal of Philosophical Logic* 1997, 26(2): 223–235; and G. Priest, "Inconsistent Models of Arithmetic Part II: The General Case", *Journal of Symbolic Logic* 2000, 65: 1519–1529.

10 The negation of Fermat's Last Theorem, or anything else can be proven just as easily, and with pretty much the same proof as the one I opened with.

explosive. That is, in a paraconsistent logic at least one proposition does not follow from a contradiction. *Ex contradictione quodlibet* is invalid according to such logics.

There are many paraconsistent logics in the market place but let me sketch the details of one, just to make the discussion concrete. The logic *LP,* is a three-valued logic with values *0*, *i*, and 1 (here 1 is "true", 0 is "false" and *i* is the other value, quite reasonably interpreted as "both true and false"). So far nothing unusual; several logics have three values. The interesting feature of *LP* is that the crucial notion of validity is defined in terms of preservation of two of the truth values: an argument is valid if whenever the truth value of the premises are not 0, the truth value of the conclusion is not 0.[11] We also need to define the operator tables for the logical connectives (i.e., define how conjunctions, disjunctions, and negations get their truth values).[12] Negation, conjunctions and disjunction (respectively) are given by the following tables:[13]

$\neg$	
1	0
i	*i*
0	1

$\wedge$	1	*i*	0
1	1	*i*	0
i	*i*	*i*	0
0	0	0	0

$\vee$	1	*i*	0
1	1	*i*	1
i	*i*	*i*	*i*
0	1	*i*	0

From these we see that if some sentence *P* has the truth value *i*, its negation, $\neg P$, also has the value *i*, and so does the conjunction of the two: $P \wedge \neg P$. Now take some false sentence *Q* (i.e., whose truth value is 0) and consider the ar-

11 This definition of validity is a natural extension of the usual definition of validity in classical logic: an argument is valid if whenever the premises are true, the conclusion is also true. The change of focus from truth to non-falsity does not matter in classical logic, since there are only two truth values (non-falsity and truth are the same thing). But in a three-valued logic, this change of focus to non-truth makes all the difference.

12 See JC Beall, and B. C. van Fraassen, *Possibilities and Paradox,* Oxford 2003; G. Priest, *Worlds Possible and Impossible: An Introduction to Non-Classical Logic,* Cambridge 2001; or G. Priest and K. Tanaka, "Paraconsistent Logic", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy* 2004, (Winter 2004 Edition), URL http://plato.stanford.edu/archives/win/2004/entries/logic-paraconsistent for full details and further discussion. The operator tables are the same as for the Kleene strong logic K_3.

13 These operator tables define negation ($\neg$), conjunction ($\wedge$), and disjunction ($\vee$) respectively. They are read as follows: (i) in the first table, read the right-hand column as giving the truth values of the unnegated proposition and the left-hand column as giving the corresponding truth value for the negation; (ii) in the second and third tables, read the top row and the left column (the ones separated from the main table by horizontal and vertical lines, respectively) to represent the truth values of the two conjuncts/disjuncts and the corresponding entry of the main table gives the truth value of the conjunction/disjunction.

gument from $P\wedge\neg P$ to Q. In *LP* this argument is invalid, since the premise $P\wedge\neg P$ does not have the truth value 0 and yet the conclusion Q does have the truth value 0. In this logic the "proof" of Fermat's Last theorem that I gave earlier is invalid.

What's the philosophical significance of all this? Well, it might just be that, mathematicians were never tempted by the above proof of Fermat's Last Theorem because the appropriate logic of mathematical proofs is a paraconsistent one. Perhaps this sounds implausible. Surely all we need to do is ask a mathematician which logic they use and surely they'll all answer "classical logic" (or perhaps "intuitionistic logic"). For various reasons it might be interesting to conduct such sociological research of mathematicians' beliefs but it will not help us answer the question at hand about the logic of mathematics. Our question is which logic do mathematicians *actually use*, and this is determined by mathematical practice, not by what mathematicians claim they use. (Indeed, most mathematicians are not experts in the differences between the various logics available.)

Perhaps, mathematician's don't use a paraconsistent logic but, rather, just avoid proofs like the five-line proof of FLT given earlier. Indeed, they might steer clear of contradictions generally. The latter is hard to do, though, when you're working in a theory that's known to be inconsistent. But perhaps part of what it takes to be a good mathematician is to recognise, not just valid proofs, but also sensible ones. On this suggestion, the proof I opened with might be formally valid but it's not sensible, since it involves a contradiction (it takes a contradiction as a premise). But this won't do as a response. First, the contradiction in question can be proven fairly straightforwardly in a very rigorous way from, what was at the time, the best available theory of sets; it's not some implausible proposition without any support. Second, not all arguments involving contradictions (or taking contradictions as premises) are defective. Take the argument from $P\wedge\neg P$ therefore $P\wedge\neg P$. Surely this is both valid and sensible. Putting these issues aside, the most serious problem with this line of response is that the notion of a sensible proof is in need of clarification. The advocate of a paraconsistent logic has no such problem here; they have only the one notion: (paraconsistent) validity and the proof in question fails to be valid.

Even if mathematicians do use classical logic but exercise some (ill-defined) caution about what proofs to accept above and beyond the valid ones, perhaps they *ought* to use a paraconsistent logic. As I've already suggested, one reason for thinking this is that the paraconsistent approach provides a more natural way to block the undesirable proofs. But there are other reasons to entertain a paraconsistent logic. There are many situations in mathematics where the consistency of a theory is called into question but without a demonstration of any inconsistency. Consider, for example, the earliest uses of complex numbers, numbers of the form $x + yi$, where $i = \sqrt{-1}$ and x and y are real numbers. There was a great deal of debate about whether it was inconsistent or just weird to entertain the square root of negative numbers.[14] Moreover, it was not just the status of complex analysis that was at issue. If the theory of complex analysis turned out to be inconsistent, everything that depended on it, such as some important results in real analysis, would also be in jeopardy. Adopting a paraconsistent logic is a kind of insurance policy: it stops the rot from spreading too swiftly and too far—whether or not you know about the rot.

Perhaps the most interesting reason to entertain a paraconsistent logic in mathematics is that with such a logic in hand, naïve set theory and naïve infinitesimal calculus can be rescued.[15] There is no need to adopt their more mathematically sophisticated replacements: ZFC and modern calculus. There are a couple of pay-offs here. First, both naïve set theory and naïve infinitesimal calculus are easier to teach and learn than their modern successors. In naïve set theory there is no need to deal with complicated axioms designed to block the paradoxes; the easily understood and intuitive unrestricted comprehension is allowed to stand. With naïve calculus there is no need to concern oneself with the subtle modern (ε–δ) definition of limit; infinitesimals are allowed back in the picture.[16] The second pay-off is related to the first and concerns the intuitiveness of the theories in question. At least in the case of set theory, the naïve theory is more intuitive. ZFC, for all its great power and acceptance, remains unintuitive and even *ad hoc*. There is no doubt that naïve set theory is the more natural theory. Similar claims could be advanced in relation to naïve infinitesimal calculus over modern calculus, though the case is not as strong here.

4. Applying Inconsistent Mathematics

I now turn to application of inconsistent mathematics. There are many interesting issues here, and I'll say just a little about a few of these. The first issue

14 See M. Kline, 1972, for some of the relevant history of this debate.

15 C. Mortensen, 1995.

16 As they are in non-standard analysis, but non-standard analysis is also rather difficult to teach and learn.

is that inconsistent mathematics adds a new twist to an old problem known as the "unreasonable effectiveness of mathematics".[17] The puzzle is to explain how an *a priori* discipline like mathematics can find applications in *a posteriori* science. As Mark Steiner puts it:

> [H]ow does the mathematician—closer to the artist than the explorer—by turning away from nature, arrive at its most appropriate descriptions?[18]

This problem has attracted the attention of physicists and mathematicians, but few philosophers have been drawn to it. Part of the reason for this is that several of the philosophers who have written on the problem seem to think that something like the following holds, and is all that's required in order to explain the puzzle.

> Mathematicians develop structures, often motivated by, or at least inspired by, physical structures. The mathematician's structures then (unsurprisingly) turn out to be similar (or even isomorphic) to various physical structures.[19]

But the fact that inconsistent mathematics, such as the early calculus, finds wide and varied applications in empirical science, raises problems for this line of thought. After all, assuming, as most of us do, that the world is consistent, how can an inconsistent mathematical theory be similar in structure to something that's consistent? There is a serious mismatch here. It certainly cannot be that the inconsistent mathematics in question is isomorphic to the world, unless one is prepared to countenance the possibility that the world itself is inconsistent. I'm not suggesting that the above thought about how to dissolve the puzzle of the unreasonable effectiveness of mathematics is completely off the mark, just that it cannot be the whole story.[20]

17 See the original paper on this, E. P. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", *Communications on Pure and Applied Mathematics* 1960, 13: 1–4, as well as M. Colyvan, "The Miracle of Applied Mathematics", *Synthese* 2001, 127: 265–278; M. Colyvan, "Mathematics and the World", in A. D. Irvine (ed.), *Handbook of the Philosophy of Science Volume 9: Philosophy of Mathematics*, North Holland forthcoming; and M. Steiner, *The Applicability of Mathematics as a Philosophical Problem*, Cambridge MA 1998.

18 M. Steiner, "The Applicability of Mathematics", *Philosophia Mathematica* 1995, 3:129–156, see p. 154.

19 See, for example, M. Balaguer, *Platonism and Anti-Platonism in Mathematics*, New York 1998, pp. 142–144, and P. Maddy, *Second Philosophy: A Naturalistic Method,* Oxford 2007, pp. 329–343, for views along these lines.

20 It is also worth noting that sometimes, when there is concern over the consistency of a mathematical theory (such as the early use of complex numbers), confidence in the theory increased when the theory was found to enjoy widespread applications.

The second issue in relation to applying inconsistent mathematics takes us into metaphysics. There is a much-discussed argument in the philosophy of mathematics known as *the indispensability argument*. This is an argument for belief in the reality of mathematical objects—Platonism—from the fact that mathematical theories are indispensable to our best scientific theories.[21] According to this line of thought, we should be committed to the existence of all and only the entities that are indispensable to our best scientific theories and, as it turns out, mathematical entities are indispensable to these theories. This leads to the conclusion that we ought to believe in the existence of mathematical entities, along with electrons, dark matter, pulsars and other entities indispensable to our best scientific theories. Again, applications of inconsistent mathematics adds a new twist. There have been times when inconsistent mathematical theories (most notably the early calculus) have been indispensable to a broad range of scientific theories. 17th and 18th century calculus was indispensable to mechanics, electromagnetic theory, gravitational theory, heat conduction and the list goes on. It seems that if one subscribes to the indispensability argument (as I do) then there's a rather unpalatable conclusion beckoning: sometimes we ought to believe in the existence of inconsistent objects.[22]

It is not clear what to make of this argument for the existence of inconsistent objects. Is it a *reductio* of the original indispensability argument? Does it tell us that consistency should be an overriding constraint in such matters? If so, on what grounds? Perhaps it is not as crazy as it sounds to believe in inconsistent mathematical objects. It is fair to say that the jury is still out on these issues, with much more work and detailed examination of case studies required before a sensible verdict can be delivered.

Finally, there has been some very interesting work on using inconsistent mathematical theories—more specifically, inconsistent geometry—to model inconsistent pictures such as those of M. C. Escher and Oscar Reutersvaard

21 See M. Colyvan, *The Indispensability of Mathematics,* New York 2001; M. Colyvan, "Indispensibility Arguments in the Philosophy of Mathematics", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy,* (Spring 2008 edition, forthcoming), URL=<http://plato.stanford.edu/archives/spr2008/entries/mathphil-indis/>; H. Putnam, *Philosophy of Logic,* New York 1971; and W. V. Quine, "Success and Limits of Matematization", in *Theories and Things*, Cambridge 1981 for details

22 M. Colyvan, "The Ontological Commitment of Inconsistent Theories", *Philosophical Studies*, forthcoming; and C. Mortensen, "Inconsistent Mathematics: Some Philosophical Implications", in A. D. Irvine (ed.), *Handbook of the Philosophy of Science Volume 9: Philosophy of Mathematics*, North Holland, forthcoming.

(e.g., Escher's Belvedere). Chris Mortensen[23] has argued convincingly that consistent mathematical theories[24] of such pictures do no do justice to the cognitive dissonance associated with seeing such pictures *as impossible*. Arguably, the dissonance arises from the perceiver of such a picture constructing an inconsistent mental model of the situation—an impossible spatial geometry. Any consistent mathematical representation of this inconsistent cognitive model will fail to capture its most important quality, namely its impossibility. Inconsistent mathematics, on the other hand, can faithfully represent the inconsistent spatial geometry being contemplated by the perceiver and thus serve as a useful tool in exploring such phenomena further. These applications of inconsistent mathematics should hold interest beyond philosophy. Indeed there are immediate applications in cognitive science and psychology. But such work is very new and the full import of it has not yet been properly appreciated.[25]

5. Conclusion

Inconsistent mathematics has received very little attention in mainstream philosophy of mathematics and yet, as I have argued here, there are several interesting philosophical issues raised by it. Moreover, some of these issues—such as the ontological commitments of inconsistent mathematical theories and the use of paraconsistent logic as the logic for mathematics—bear directly on contemporary debates in philosophy of mathematics. Other issues—such as the application of inconsistent mathematics to model inconsistent pictures—promise to take philosophy of mathematics in new and fruitful directions. For my money, though, the biggest issue concerns possible insights into the relationship between mathematics and the world. This is a central problem for both philosophy of mathematics and philosophy of science.

I believe that there is a great deal to be learned about the role of mathemati-

23 C. Mortensen, "Peeking at the Impossible", *Notre Dame Journal of Formal Logic*, 38(4): 527–534; C. Mortensen, "Inconsistent Mathematics", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy,* (Fall 2004 edition), URL=<http://plato.stanford.edu/archives/fall/2004/entries/mathematics-inconsistent/>; and C. Mortensen forthcoming.

24 Such as in L. S. Penrose and R. Penrose, "Impossible Objects, a Special Kind of Illusion", *British Journal of Psychology* 1958, 49: 31–33; and R. Penrose, "On the Cohomology of Impossible Pictures", *Structural Topology* 1991, 17: 11–16.

25 Although see C. Mortensen forthcoming.

cal models—both consistent and inconsistent—in scientific theories, by paying closer attention to the use of inconsistent mathematics in applications. Perhaps focussing our attention on the consistent mathematical theories has misled us to some extent. If this is right, we won't have the complete picture of the mathematics–world relationship until we understand how inconsistent mathematics can be so useful in scientific applications.[26]

26 I'd like to thank Stephen Gaukroger and Audrey Yap for helpful conversations on the history of the calculus, and Adam La Caze and Fabien Medvecky for comments on an earlier draft. I have also benefited from several conversations with Chris Mortensen about inconsistent mathematics. Work on this paper was funded by an Australian Research Council Discovery Grant (grant number DP0209896).