The Ins and Outs of Mathematical Explanation

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Gauss referred to mathematics as the queen of sciences. His remark was intended to flag the privileged and exalted position mathematics occupies within the sciences. And just as royalty is set apart from its subjects, so too is mathematics set apart from the rest of science. Mathematics is certain, its results stand for all time, and it proceeds by a priori methods. The rest of science, by contrast, is uncertain, fallible, and a posteriori. In what follows I'll highlight another interesting difference between mathematics and the rest of science. This difference concerns the way explanation operates in mathematics and the surprising way mathematical explanations can feature in broader scientific explanations.

First a bit of background on scientific explanation. There are a number of important questions that naturally arise about explanation. When is an explanation required? How do we recognise explanations as explanations? What is an explanation? There are no easy answers to any of these questions but for present purposes we can take partial answers to the first two of these questions to be that an explanation is called for when an appropriate "why" question is pressing. Why did the car skid off the road? Why did the the anode emit x-rays? An explanation, in its most general sense is an answer to such "why" questions and in answering such questions, explanations reduce mystery. Exactly how an explanation reduces mystery brings us to the third of the questions above. There are many competing philosophical accounts of explanation and I won't try to do justice to them here. By way of example, I'll mention just one. The causal account of explanation holds that an explanation is the causal history of the event in need of explanation. Once the causal history is presented, the event in question is no longer thought to be mysterious (e.g. because the posterior probability of the event, conditional on the causal history, is higher than the unconditional prior probability of the event in question). Now I turn to mathematical explanation.

Intra-Mathematical Explanation

It is well known that some but not all proofs are explanatory. Explanatory proofs tell us *why* the theorem in question is true, whereas the non-

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explanatory proofs merely tell us *that* the theorem in question is true [10]. With an explanatory proof we have one mathematical fact being explained by another (or other) mathematical fact(s). Call such cases of explanation within mathematics *intra-mathematical explanations*.

An example will help. Consider the well-known Euler result that there is no way of traversing the seven bridges of Königsberg once and only once in a single trip, beginning and ending in the same place.

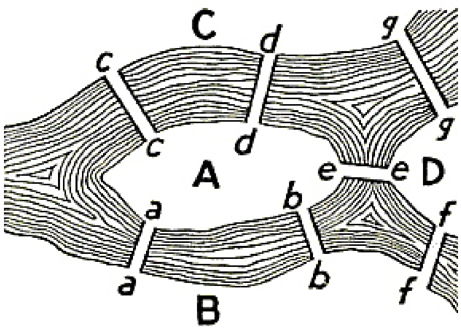
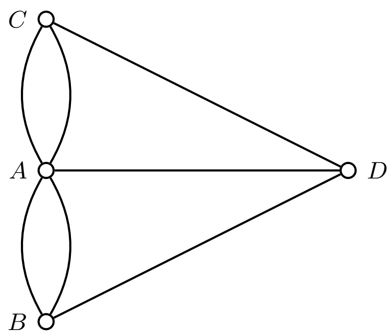


Figure 1: *The seven bridges over the River Pregel in Königsberg (in Euler's time)* [17]

In its mathematical formulation this becomes: there is no Eulerian cycle for the following multigraph.



The crucial result here is that a connected graph has an Eulerian cycle iff it has no vertex with an odd degree [16]. That is, there are no vertices (or nodes) with an odd number of edges meeting there.[1] The central idea of the graph-theoretic proof is that on any Eulerian cycle, every arrival at a vertex must be accompanied by a departure; an odd valence for a vertex signifies an arrival without a corresponding departure or a departure without a corresponding arrival. Indeed, this is the core explanatory insight of the graph-theoretic proof.

[1] Or to revert to the original problem: the walk is possible so long as it does not involve an odd number of bridges with one end on the same land mass.

Compare such a proof with a brute force, combinatorial proof that there is no Eulerian cycle for the above multigraph. The latter would indeed deliver the result, but armed only with such a proof, we would be none the wiser as to why there is no Eulerian cycle for the multigraph in question. We'd just know that all options had been tried and none of them worked. Moreover, the graph-theoretic proof is more general. It shows, for example, that knocking out one of the edges from C to A will not help. The brute force method would need to start over again to show that there is no Eulerian cycle for the modified graph.[2]

What is the sense of explanation at work here? It seems rather different from explanations elsewhere in science and in everyday life. When we explain why the car accident occurred, we appeal to driver fatigue, the icy road, the velocity of the vehicle and so on. In short, the explanation provides the causal history of the accident: all the temporally-prior events or conditions that ultimately resulted in the accident. But mathematical explanation is not like this. Talk of causes is completely out of place in mathematics. The existence of the number i *caused* the polynomial $X^2 + 1$ to have two roots? No! The existence of i is the reason that $X^2 + 1$ has two roots but it is a category mistake to think of i causing anything. Numbers and other mathematical objects do not seem to be the right kind of things to be in the causal nexus of the universe.[3]

It might be tempting to suggest that explanatory proofs don't provide causal explanations but something stronger: deductive explanations. After all, the conclusion of a chain of deductive reasoning, as we have in the proof of a mathematical theorem, follows of necessity. It doesn't merely follow because of contingent events in the past. While this is right — mathematical consequence is typically deductive and this is stronger than causal consequence — deduction cannot be the key to mathematical explanations. If it were, all proofs would be explanatory, but this is clearly not the case. For example, the brute force, combinatorial proof mentioned above also delivers a deductive result. But, as we have seen, this proof is not explanatory. The source of mathematical explanation must lie elsewhere.

Perhaps mathematical explanation relates to the structure of the proof.[4] We have a variety of different proof structures in mathematics: For example,

[2]The graph-theoretic proof explains the *mathematical fact* that there is no Eulerian cycle for the above multigraph. That's the intra-mathematical explanation arising from the explanatory proof. But the proof also plays a crucial role in explaining why the previously-described walk around Königsberg cannot be completed. The latter is the related extra-mathematical fact. I'll have more to say about this quite different kind of explanation shortly.

[3]Indeed, this is one of the reasons many philosophers have misgivings about the existence of mathematical objects. The causal inertness of mathematical objects makes it difficult to understand how we can have mathematical knowledge [3].

[4]There is also an interesting question about the status of pictures in proofs: can pictures serve as proofs and do they, in some cases, deliver genuine understanding [4]?

there is conditional proof, reductio ad absurdum, finite induction, transfinite induction, disjunctive syllogism, universal generalisation, and proof by cases, to name a few. But there are problems with the view that the explanatoriness of a proof rests entirely on its structure. Apart from anything else, it is often relatively easy to transform the proof structure without any substantial change to the main body of the proof. Think, for example, of how, with a small change in the set up, a reductio proof of the infinitude of the primes can be turned into a proof by universal generalisation [7].

A more promising strategy is to look below the level of structure of the proof and at the details of the proof. Here we find two quite distinct lines of thought arising. One is that a proof is explanatory because it proceeds via the "right kind" of paths. What are the right kind of paths? Perhaps they are paths that connect results to other results in the same domain of mathematics. This may be why elementary proofs are valued in number theory: they deliver number-theoretic results via number-theoretic means, without excursions into complex analysis. On the other hand, sometimes a proof is seen to be explanatory because it builds bridges between different areas of mathematics, often by showing that a result in one area is a special case of a more general result. Category theory proofs of results in group theory (such as the Free Group Theorem) may be instances of this [14, p. 123]. If this is right, there are two quite distinct kinds of explanation operating in mathematics: one local and the other more global and unifying.

Extra-Mathematical Explanation

So far we have been considering the explanation of one mathematical fact by another. Contrast this intra-mathematical explanation with another kind of explanation in which mathematics can feature: the explanation of some physical phenomenon by appeal to mathematics. Call the latter *extra-mathematical explanation*. This is more controversial. Although our understanding of intra-mathematical explanation leaves a lot to be desired, its existence is widely accepted. But there is a rather heated philosophical debate surrounding the very existence of extra-mathematical explanation.[5]

What's all the fuss about? The main reason for some philosophers objecting to extra-mathematical explanation is that it would seem to commit them to Platonism: the view that mathematical objects such as sets, numbers, functions, and the like exist. Why? If you subscribe to a principle known as *inference to the best explanation*, you're committed to all the entities that play an explanatory role in your best science: electrons, quarks, black holes, gravitational waves, and so on. Add mathematical objects to this list and it just gets too weird for some philosophers' tastes. Unless they

[5]The main proponents of extra-mathematical explanation are Alan Baker [2], Mary Leng [12] and the present author [5, 6].

are willing to give up on the principle of inference to the best explanation, those who find Platonism unacceptable are forced to deny that there are any extra-mathematical explanations. Be that as it may, there seems to be no shortage of examples of mathematics apparently playing crucial explanatory roles in science. Here I'll quickly look at a couple of these.

The first involves the Honeycomb Theorem. Charles Darwin observed the incredible regularity of hexagons in honey bee hives and conjectured that the hexagonal shape was in some sense optimal [8]. Darwin was right but vindication took some time. It finally came in 2001 from a mathematical theorem: the honeycomb theorem [11]. The Honeycomb theorem states that a hexagonal grid represents the best way to divide a surface into regions of equal area with the least total perimeter. Of course the mathematics on its own doesn't explain why honey bees build hexagonal hives. We also need some biology: wax is expensive so bees need to minimise its use (hence minimising total perimeter) while maximising area for honey storage (there's no wasted space in the hive). It's also worth noting that this is a tiling problem not a sphere packing problem. This is because bees need access to the cells. Now throw in some evolutionary theory to explain why less efficient bees have been selected against and we are left with hexagonal hives. Although the full explanation involves a mixture of biology and mathematics, it's the mathematics that's doing the heavy lifting [13].

Another phenomenon whose explanation involves mathematics is the life-cycle of the Magicicada. There are two subspecies of North American cicadas known as *Magicicada* with prime number life cycles. These cicadas appear from the ground *en masse* once every 13 and 17 years respectively. The explanation biologists offer for these unusual life cycles is in terms of evolutionary advantage. A periodic organism trying to avoid a periodic predator needs to minimise the number of years of overlap between itself and the predator. It is rather straightforward to prove that having a prime life cycle does this [2]. If a life cycle has a prime period, predators need very specific periods to overlap on a regular basis. For example, the predator would need the same prime life cycle, suitably coordinated. Again we see that the explanation relies on mathematics — in this case some elementary number theory. Moreover, the mathematics seems to be doing the important work in the explanation.

This last case also highlights an interesting general point about extra-mathematical explanations. These explanations typically tell us that not only is the world thus and so but, in a very important sense, it *had to be thus and so*. The cicada life cycles are apparently squeezed from above by (not well-understood) biological considerations and from below by the fact that small primes are not as effective for predator avoidance. The remaining window, arguably, has 4 primes: 7, 11, 13 and 17. The periods found in nature are the largest primes in this set. These represent the optimal predator avoidance strategies, subject to various biological constraints.

There are many other cases of such extra-mathematical explanations discussed in the literature.[6] What is common to them all is that the mathematics is crucial to the success of the explanation and if there is any mathematics-free alternative available, such explanations are impoverished in various ways. In particular, the mathematics-free, causal explanations get bogged down in the contingent detail and fail to reveal the big picture. For example, a causal account of the Kirkwood gaps in the asteroid belt would give the causal histories of all the asteroids in the vicinity and would show why each asteroid fails to be orbiting in one of the Kirkwood gaps. Without the mathematical analysis, it looks as though it's a mere coincidence that there are no asteroids in the Kirkwood gaps. But it is no coincidence. The Kirkwood gaps are unstable orbits and this crucial piece of the story is delivered by the mathematical explanation [7, chap. 5]. Part of the power of mathematics is that it enables abstraction away from the often-irrelevant, contingent details and goes straight to the core of the explanation.

What is Explanation?

We have seen that mathematics can deliver both intra- and extra-mathematical explanations. This is important for a number of reasons. First, the usual accounts of explanation one finds in the philosophy of science do not readily accommodate either of these kinds of mathematical explanation. Second, neither kind of mathematical explanation is causal, so popular causal theories of explanation are at best one species of explanation. Third, the existence of extra-mathematical explanation would seem to lend support to Platonism.

The relationship between these two kinds of mathematical explanation requires further investigation. Philosophers of mathematics have been largely interested in extra-mathematical explanation and mathematicians have long been interested in intra-mathematical explanation, but the ease with which we accept the application of intra-mathematical explanations to physical phenomena, as in the Honeycomb Theorem case, suggests that these two kinds of mathematical explanation may be closely related. Philosophers of mathematics and mathematicians really ought to hang out more. For a start, it would be extremely useful for philosophers to have a good stock of proofs considered by mathematicians to be explanatory and proofs not considered to be explanatory.[7] It would also be good to have mathematicians' thoughts on what distinguishes explanatory proofs from the others. It

[6] Other examples are the Euler graph-theory explanation of why the bridges of Königsberg walk cannot be completed, why the physical act of squaring the circle is impossible, and why the Kirkwood gaps in the asteroid belt have the specific locations they do. See [15] for discussion of some of these examples.

[7] Some steps towards this goal have been made in [1, 9].

would be interesting to explore whether proof is the only locus of explanation. After all, if one kind of intra-mathematical explanation lies in unifying branches of mathematics, perhaps domain extensions and even generalised definitions could be thought to facilitate explanation. These are all issues on which this philosopher of mathematics would welcome the opinions of mathematicians.[8]

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