MICHAEL D. RESNIK
Mathematics as a Science of Patterns
Oxford, Clarendon Press, 1997, cloth, £37.50

STEWART SHAPIRO
Philosophy of Mathematics: Structure and Ontology
New York, Oxford University Press, 1997, cloth, £40.00

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Structuralism in the philosophy of mathematics is the view that the proper subject matter of mathematics is the relationships between various kinds of mathematical entities, rather than the entities themselves. So, for instance, number theory is the study of ω-sequences. The nature of the entities that constitute the ω-sequences is irrelevant—what is important is the *structure* that is common to all ω-sequences. There is something undeniably right about structuralism. Indeed the structuralist slogans, "mathematical objects are places in structures" or, Michael Resnik's favourite, "mathematics is the science of patterns", seem to reveal important insights into the nature of mathematics and its subject matter.

The real significance of structuralism, though, is in its ability to provide answers to some rather difficult problems in the philosophy of mathematics. For example, structuralism is able to explain why mathematicians are typically only interested in describing the objects they study up to isomorphism—for that is all that there is to describe. Structuralism is also able to explain why we're inclined to think that either all of the natural numbers exist or none of them exist—for the natural numbers, like other structures, come as a package. Most importantly, structuralism provides a neat and plausible response to both Frege's Julius Caesar problem (Cf. 'The Concept of Number' in P. Benacerraf and H. Putnam, *Philosophy of Mathematics: Selected Readings*, 2nd Edition, Cambridge University Press, 1983) and Paul Benacerraf's underdetermination problem ('What Numbers Could Not Be', also in P. Benacerraf and H. Putnam). According to structuralists, Julius Caesar *can* play the role of the number two, so long as Caesar has the appropriate relationships with other entities, and together they make up an ω-sequence (with Caesar being the successor of the successor of the entity playing the zero role, of course). Similarly, either von Neumann's {Ø,{Ø}} or Zermelo's {{Ø}} can play the role of the number two. According to the structuralist, however, neither Julius Caesar nor {Ø,{Ø}} nor {{Ø}} *is* the number two, for the number two is no more and no less than a position in a structure.

Structuralism has had its fair share of supporters over the years too. These have included mathematicians of the calibre of Dedekind, Hilbert and Poincaré, and philosophers such as Quine and Benacerraf and (somewhat tentatively) Putnam. Few, however, have provided detailed defences of this position and even fewer have given clear articulations of exactly what it is that structuralists are committed to. It is for these reasons that Michael Resnik's and Stewart Shapiro's books are very welcome additions to the literature in the philosophy of mathematics. Resnik and Shapiro are two of the most prominent contemporary defenders of this important position and it is great to see their views collected together in monographs.

Despite its undeniable appeal and influential supporters, structuralism has not been without its problems. It's probably not too gross a generalisation to say that the main problems that have faced structuralism have been concerned with lack of clarity. After all, the slogans used to describe the view are nothing but highly evocative metaphors. In particular, philosophers have wondered: what is a structure?; by virtue of what do two systems display *the same* structure?; what is the ontological status of the structure itself?; what is the ontological status of the positions in the structure?; is structuralism a realist or anti-realist philosophy of mathematics?; how do we have epistemic access to (apparently abstract) structures? All of these are important questions and ones which need to be properly addressed before structuralism can be taken to be a fully-fledged philosophy of mathematics. A great deal of the two books in question are devoted to answering these questions. Indeed, as we have come to expect from both Resnik and Shapiro, the discussions of these questions are clear, thoughtful, and for the most part (I found), rather convincing.

There is a great deal of similarity between Resnik's and Shapiro's views, so many of the above questions are answered along similar lines: both Resnik and Shapiro are mathematical realists—they both believe that mathematical entities exist independently of our knowledge of them and that mathematical statements are mind-independently true or false; they both agree that structuralism and realism are independent of one another; and they both provide essentially the same account of sameness of structure. Moreover, they both agree that a structuralist *can* make sense of identifying positions from different structures. (It is often claimed that structuralists cannot identify the natural number 2 with the real number 2 and the complex number $2 + 0i$, since these are positions in *different* structures.) Resnik and Shapiro agree that there is good reason to think that 'two' refers to the same object irrespectively of the background structure. Shapiro claims that this is a matter of *stipulation*—there is no fact of the matter about trans-structural identifications.

Given so much agreement, it is worth noting that one of the issues on which Resnik and Shapiro differ is that old platonist bugbear: epistemology. Resnik's epistemology is essentially Quinean. He argues for mathematical realism by explicit appeal to indispensability considerations and employs the familiar Quinean doctrines of naturalism and holism. This is not to say, however, that Resnik has nothing to add to Quine's well known views on how we come to gain knowledge of abstract mathematical entities. Resnik's attention to detail and sensitivity to the various arguments used to cast doubt on Quine's position has led to various clarifications, refinements and innovations. For instance, Resnik provides a new "pragmatic" indispensability argument which appears to be quite distinct from the traditional Quine-Putnam version. Moreover, Resnik's argument seems to avoid some of the recent attacks on the traditional argument and as such it deserves careful attention by mathematical realists and anti-realists alike.

As with Quine, Resnik *postulates* entities. He recognises that such an approach appears to sit a little uneasily with realism:

> Postulational approaches seem better suited to conventionalists, who may claim that we make truths, than to realists, who must hold that we can only recognize independently obtaining truths. (Resnik, p. 175)

He begins his attempt to quell this unease by distinguishing two different questions:

> [W]e must separate the question of whether [our mathematical ancestors] were justified in *positing* mathematical objects from the question of whether they were justified in *believing* in the things they posited. (Resnik, p. 185)

The crucial question for the realist, of course, is the question about belief. Not surprisingly, Resnik claims that (most) beliefs about mathematical posits *are* justified, and they're justified by the role they play in simplifying and unifying both mathematical theory and our best physical theories. This marks the difference between mathematical posits and the posits of literary fiction. In short, we know about mathematical entities because we posit them and such positing helps simplify and unify our best theories.

Shapiro's epistemology, on the other hand, relies on less familiar devices (in epistemology at least): *abstraction* and implicit definition. At the lowest level he takes it as given that we are able to recognise structures, for otherwise we could not recognise all the different tokens of the English letter 'E', say. Moreover, such recognition involves abstraction. Recognising natural-number structures is in principle no different. Shapiro claims that this certainly does not provide an account of mathematical pattern recognition, but the phenomenon is no more mysterious than the general problem of pattern recognition: "[i]n the end, we either demystify numbers or we mystify more mundane items." (Shapiro, p. 114)

Appeal to simple abstraction and similar techniques is limited—it will not give us cardinalities large enough for the higher reaches of set theory. Such cardinalities, however, are provided by a more "speculative and problematic" (Shapiro, p. 129) technique. Shapiro argues that

> grasping a [mathematical] structure and understanding the language of its theory amount to the same thing. There is no more to understanding a structure and having the ability to refer to its places than having the ability to use the language correctly . (Shapiro, p. 137)

Thus, for Shapiro, "language provides our epistemic access to mathematical structures." (Shapiro, p. 137)

My portrayal of Resnik's and Shapiro's respective epistemologies is obviously rather superficial, but I hope not misleading because of this. What I find interesting about their discussions here is that despite a great deal of consensus on other matters, epistemological concerns drive Resnik to a fairly modest realism—enough mathematical objects to service empirical science (and a bit more). Whereas Shapiro is driven to what may seem like a rather extravagant position in which every coherent structure is taken to be real. Although my own sympathies lie more with Resnik than with Shapiro, this is not the place to take up that issue. It is clear, however, that the role of epistemology in the philosophy of mathematics did not decline with the decline of the causal theory of knowledge!

So far I have been concerned largely with Resnik's and Shapiro's commitment to structuralism. Although structuralism is certainly the central thesis of both books, I do not wish to give the impression that that is *all* these two books are about. Both books touch on many interesting and important topics to which both authors make major contributions. For example, both books provide enlightening discussions of truth, realism and anti-realism: Michael Resnik argues that mathematical realists need only be committed to an *immanent* conception of truth; and Shapiro provides a very nice survey of the various anti-realist positions in the philosophy of mathematics that try to reduce ontology by invoking modality. Of course it's not possible to discuss these topics in any detail here, but I *do* wish to draw attention to the fact that both books make significant efforts to connect their central theses with other topics both within the philosophy of mathematics and further afield.

In summary, these two books are interesting, well written, and go a long way to alleviate the concerns that some philosophers have with mathematical structuralism. At the very least, Resnik and Shapiro have both provided clear, well motivated accounts of their own influential versions of structuralism, and so these two books will no doubt serve as convenient starting points for all future discussions of structuralism in the philosophy of mathematics. *Mathematics as a Science of Patterns* and *Philosophy of Mathematics: Structure and Ontology* are thus timely and important additions to contemporary philosophy of mathematics.