Mathematics: Truth *and* Fiction?

MARK BALAGUER. ***Platonism and Anti-Platonism in Mathematics.*** New York: Oxford University Press, 1998. Pp. x + 217. ISBN 0-19-512230-5

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Mark Balaguer's project in this book is extremely ambitious; he sets out to defend *both* platonism and fictionalism about mathematical entities. Moreover, Balaguer argues that at the end of the day, platonism and fictionalism are on an equal footing. Not content to leave the matter there, however, he advances the anti-metaphysical conclusion that there is no fact of the matter about the existence of mathematical objects.[1]

Despite the ambitious nature of this project, for the most part Balaguer does not shortchange the reader on rigor; all the main theses advanced are argued for at length and with remarkable clarity and cogency. There are, of course, gaps in the account (some of which are described below) but these should not be allowed to overshadow the significant steps Balaguer takes towards an extremely interesting philosophy of mathematics—a philosophy of mathematics for which the present reviewers have considerable sympathy. In short, this book is an important contribution to the philosophy-of-mathematics literature.

1. Balaguer's Project

The book is divided into three parts. In the first, Balaguer tries to show that no good arguments have been advanced against what he argues is the best version of platonism. More specifically, he defends what he calls *full-blooded platonism* ('FBP'), the view that every mathematical object that could possibly exist, does exist. It is important to the conclusions later in the book that FBP is the only viable form of platonism, so in this first section Balaguer also attempts to demonstrate that all other platonist positions are indefensible.

In the second part of the book, Balaguer tries to show that no good arguments have been advanced against (a broadly Fieldian kind of) fictionalism. Although it is *fictionalism* that Balaguer defends, he also makes

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[1] This somewhat speculative and certainly more controversial conclusion has resonances in the recent work of Penelope Maddy [1997].

it clear that other anti-realist positions, such as deductivism and formalism, are more or less equivalent to fictionalism and so he has no serious quarrel with them. He prefers fictionalism, however, because it 'provides a standard semantics for the language of mathematics' (p. 104), whereas other anti-realist accounts (such as Chihara [1990], for instance) need to reinterpret mathematical discourse.

Finally, in the third part, Balaguer discusses the consequences of his defense of both platonism and fictionalism. It is in this part that he advances the anti-metaphysical thesis mentioned earlier and proposes what he calls 'a kinder, gentler positivism' (p. 159). Let us now outline each of these three parts of Balaguer's project in more detail.

1.1 Defending Platonism

Undoubtedly the biggest obstacles to a platonist philosophy of mathematics are the problems Benacerraf discusses in [1973] and [1965], respectively: (i) the problem of providing a naturalized epistemology and (ii) the non-uniqueness problem. Consider the first of these. In [1973], Benacerraf challenged platonists to provide an account of how we come by knowledge of abstract mathematical entities that is consistent with knowledge acquisition in other domains. Although Benacerraf explicitly formulated the problem in terms of the causal theory of knowledge, his central concern can be separated from this problematic epistemology (see Field [1989], pp. 25–26).

Balaguer discusses most of the standard attempts by platonists to meet this challenge and finds them all wanting. He then shows how one version of platonism, namely FBP, *can* meet the Benacerraf challenge. The basic idea here is quite simple. Balaguer notes, with Hartry Field ([1989], pp. 26–27), that it would be rather mysterious if someone had true beliefs about the day-to-day events in a remote village in Nepal, without any mechanism explaining the correlation between the person's beliefs and the events in the village in question. But such, it seems, is the plight of the platonist; they would have us believe that we have true beliefs about an abstract realm with which we do not (and *can* not) have any causal contact. But as Balaguer points out:

> [I]f all possible Nepalese villages existed, then I *could* have knowledge of these villages, even without any access to them. To attain such knowledge, I would merely have to dream up a possible Nepalese village. For on the assumption that all possible Nepalese villages exist, it would follow that the village I have imagined exists and that my beliefs about this village correspond to the facts about it. (p. 49)

Of course not all possible Nepalese villages exist, and so such an epistemology for Nepalese villages is rather unpromising. Balaguer's point, however, is that on the assumption that all possible Nepalese villages *do* exist, there is no mystery about how our beliefs about Nepalese villages constitute knowledge.

Balaguer now extends this point to the case of platonism. Benacerraf's epistemological challenge for platonism can be met on the assumption that every mathematical object that could exist, does exist (as FBP maintains). According to FBP every consistent mathematical theory describes *some* part of the mathematical realm. So our beliefs (*via* axioms or correct inference) about the mathematical objects of a consistent theory constitute knowledge of those objects.

It is a (literally) stunning and yet intriguing idea to argue that *increasing* one's ontology to the limit[2] can solve the platonist's epistemological problem. There are details to be tidied up of course. Note, for example, a certain tension in the above passage from Balaguer—if each possible Nepalese village is a complete and determinate village, how does Balaguer justify talk of '*the* village I have imagined' on the basis of an episode of imagination that, at best, yields an incomplete description of a Nepalese village?[3] There is also the question of how we can know about the consistency of mathematical theories. Balaguer does attempt to address these and other details, but we won't pause over them at this point.

The fact that FBP offers a solution to the epistemological challenge to platonism now becomes a positive argument for favoring it over other versions of platonism: FBP is (allegedly) the only platonist position able to deal with Benacerraf's epistemological challenge. The other motivations for FBP concern the ways in which it meshes with standard mathematical practice. For example, Balaguer claims that FBP 'reconciles the objectivity of mathematics with the extreme *freedom* that mathematicians have' (p. 69).

After demonstrating how FBP meets Benacerraf's epistemological challenge to platonism and providing some motivation for FBP, Balaguer then (Chapter 4) defends FBP against Benacerraf's other major problem for platonism: the non-uniqueness problem. In [1965], Benacerraf noted that there are an infinite number of equivalent and equally effective ways to reduce simple number theory to set theory. He concluded that numbers could not be sets because there is no non-arbitrary way to identify them as particular sets. (Benacerraf went on to conclude that numbers aren't even objects, since *all there is* to 'the numbers' are the structural relations defined by the axioms of number theory.) Balaguer reconstructs this into a general argument against platonism, since it is traditionally a part of platonism that mathematical theories (and, in particular, number theory) are about *unique* collections of mathematical objects. As Balaguer formulates it, the argument starts with the uncontroversial premises that there are numerous

[2] Well not quite to the limit. Balaguer does not consider mathematical theories developed in the context of paraconsistent logic, such as those discussed by Chris Mortensen [1995] and Graham Priest [1997].

[3] See Cheyne [1999] for criticism of Balaguer's project in this regard.

sequences of abstract objects that satisfy the axioms of number theory and that there is nothing 'metaphysically special' about any of these sequences. The intermediate conclusion is that there is no unique sequence of abstract objects that is the natural numbers. So if platonism were to entail that there is a unique such sequence, one could reach the final conclusion that platonism is false.

Balaguer's solution here (pp. 84–91) is to argue that FBP is *not* committed to the idea that our theories describe unique collections of mathematical objects. FBP is committed to the existence of *all* the mathematical objects that could possibly exist. Thus FBP is committed to all the mathematical objects in all the models of the Peano-Dedekind axioms—intended and unintended alike. (Similarly, FBP is committed to the existence of ZFC sets, non-well-founded sets, ZFC + $V = L$ sets, ZFC + $V \neq L$ sets and so on.) Balaguer argues, however, that we can take arithmetic as being about only the models that are intended. Again, there are details to be tidied up and some nagging questions (which we defer until later), but to his credit, Balaguer recognizes these and attempts to address them, concluding that non-uniqueness is a non-problem for FBP (which is also the conclusion of Balager [1998]).

1.2 Defending Fictionalism

Now we turn to the second part of Balaguer's project: defending fictionalism about mathematical objects. First he argues that most anti-realist philosophies of mathematics are not significantly different from fictionalism and that none of these non-fictional anti-realist accounts has any advantage over fictionalism.

> The various versions of anti-realistic anti-platonism do not differ from one another in any metaphysical or ontological way, because they all deny the existence of mathematical objects. ...They differ only in the interpretations that they provide for mathematical theory and practice. But as soon as we appreciate this point, the beauty of fictionalism and its superiority over other versions of anti-realism begins to emerge. For whereas fictionalism interprets our mathematical theories in a very standard, straightforward, face-value way, other versions of anti-realism—such as deductivism, formalism, and conventionalism—advocate controversial, non-standard, non-face-value interpretations of mathematics that seem to fly in the face of actual mathematical practice. (p. 102)

(No doubt defenders of other non-fictional anti-platonist philosophies of mathematics will be quick to point out that their accounts don't have to ascribe systematic error to mathematicians when mathematicians assert that sentences such as 'there is an even prime number' and '$2^\kappa > \kappa$' are true. This is one way in which fictionalism clearly *does* fly in the face of standard mathematical practice. Balaguer does address this worry (p. 100), but in any case we can set the issue aside, for ultimately Balaguer's talk of 'fictionalism' can be construed very broadly to include *all* the relevant versions of anti-realistic anti-platonism. Nothing of any significance hangs on Balaguer's preference for Field-style fictionalism.)

Balaguer sets out to address the big problem for fictionalism, namely, the Quine-Putnam indispensability argument. Quine [1948] and Putnam [1971] have argued that we ought to be committed to all the entities indispensably quantified over in our best scientific theories; amongst these entities, claim Quine and Putnam, are various mathematical entities. Hartry Field has responded to the Quine-Putnam argument by denying that quantification over mathematical entities is indispensable to science. In particular, he has given an account of how mathematical theories (which are, according to fictionalists, strictly speaking, false) can be used in our best science.[4] Moreover, Field has begun work on the enormous task of nominalizing science. Field's project has attracted a great deal of criticism, of which perhaps the most significant is that it is hard to see how the approach he adopts in [1980] can be applied to quantum mechanics.[5]

Balaguer clearly has a great deal of sympathy with Field's project, despite the fact that at the end of the day the approach Balaguer adopts to the indispensability argument does not depend on the success of Field's project. (Balaguer does, however, devote a chapter to outlining how one might go about nominalizing quantum mechanics.) Balaguer's approach to the indispensability argument is to argue that mathematical entities, because of their lack of causal powers, could not make a difference to the way the *physical* world is. This leads him to defend the position he calls 'nominalistic scientific realism', namely, the view that the content of our scientific theories can be separated into nominalistic and platonistic components and that the nominalistic content (*i.e.*, the purely physical facts described by such theories) is true (or mostly true), while the platonistic content (*i.e.*, the abstract mathematical facts described by such theories) is fictional (p. 131). Thus, even if mathematics is indispensable to science, there is no reason to believe anything other than the nominalistic content of our scientific theories.

Balaguer illustrates his view by considering sentences such as:

(A) The physical system S is forty degrees Celsius.

He argues that while (A) does assert that a certain relation holds between S and the number 40, fictionalists can maintain that since the number 40 is causally inert, the truth of (A) depends on purely nominalistic facts about S and purely platonistic facts about the natural numbers; Balaguer argues that these two sets of facts hold or don't hold independently of one an-

[4] This is because the mathematized science is a *conservative extension* of the nominalistic scientific theory.

[5] See Malament [1982] for the details of this objection.

other. Thus fictionalists can maintain that facts of the one sort obtain, whereas facts of the other sort do not, *i.e.*, that the nominalistic content of (A) is true whereas its platonistic content is fictional. Moreover, Balaguer argues that fictionalists can take this view of the whole of empirical science, maintaining that while empirical science is not true (because there are no abstract objects), the nominalistic content of empirical science is true (because, as he puts it, 'the physical world holds up *its end* of the "empirical-science bargain" ' (p. 134)). He also argues that this is a sensible view, and a genuine form of scientific realism, because 'the nominalistic content of empirical science is all empirical science is really "trying to say" about the world' (p. 141).[6] We will discuss Balaguer's argument to this conclusion in a little more detail in Section 2 of this review.

1.3 The Anti-Metaphysical Conclusion

The conclusions of the first two sections of Balaguer's book (if correct) tell us that neither fictionalism nor (full-blooded) platonism has any clear advantage over the other. In the final section, Balaguer suggests that there will *never* be any good argument to settle the question of the existence of mathematical entities. He then goes on to argue that there is no fact of the matter about the existence of mathematical entities. Although the conclusions in the third section are advanced with less confidence than those of the previous two sections,[7] Balaguer once again provides rather interesting arguments for the ultimate conclusion of the book: fictionalism and FBP are *both* correct with regard to everything but ontology; with regard to ontology, *neither* is correct (p. 179).

Balaguer begins by arguing that both FBP and fictionalism about mathematics 'share the same "vision" of mathematical practice' (p. 157). In support of this he draws attention to the many points of agreement between these seemingly opposed philosophical positions. One worth mentioning here is that both fictionalists and FBPists agree that all consistent, mathematical theories are ontologically on a par—for the fictionalist they are all false, for the FBPist they are all true. He argues that the only thing the two views *do*, and in fact *could*, disagree on is ontology. But the lack of causal powers of mathematical entities ensures that there is no way of choosing between the two views in question.

This stronger epistemological claim (that we could never know whether FBP or fictionalism is the correct metaphysical account of mathematics) does not, of course, establish the anti-metaphysical conclusion which Balaguer seeks (there is no fact of the matter about whether mathematical entities exist). This latter conclusion, Balaguer argues, *does* follow once

[6] Balaguer's views here are very similar to those of Nancy Cartwright [1983] on physical laws and the view advanced by Jody Azzouni in [1997] on abstract entities.

[7] The anti-metaphysical conclusion, in particular, Balaguer sees as a 'first shot' at an argument in this direction (p. 158).

one considers the possible-worlds-style truth conditions for sentences such as

(*) There exist abstract objects; that is, there are objects that exist outside of spacetime (or more precisely, that do *not* exist *in* spacetime). (p. 159)

He suggests that:

> We don't know what existence outside of spacetime would be *like*, and so we don't know what the possible-worlds-style truth conditions of (*) are, and therefore our usage doesn't determine what these truth conditions are. But since (*) is *our* sentence, it could obtain possible-worlds-style truth conditions only from our usage, and so it follows that (*) simply doesn't *have* any such truth conditions. (pp. 171–172)

Balaguer goes on to argue that this implies that there is no fact of the matter about the truth or falsity of (*). He thus supports a position that is somewhat positivist in flavor. But as Balaguer points out, his view is 'kinder' than positivism in that it is not directed at all metaphysical debates; it is 'gentler' in that it does not deny that sentences about the existence of mathematical entities are meaningful (p. 159).

2 Critical Discussion

In a book that covers as much territory and defends as many controversial theses as this, one expects that there will be scope for disagreement. Though we have a high regard for this book and are sympathetic to Balaguer's conclusion concerning the standoff between platonism and fictionalism, there are (aspects of the) arguments that Balaguer puts forward in reaching this conclusion that need more work. In what follows, we try to briefly indicate these.

(1) The versions of platonism and fictionalism that Balaguer defends are not formulated with a high degree of precision—only a sketch of these theories is offered. Given the rough formulation of full-blooded platonism (p. 7), it is difficult to see how one could derive the platonic truth conditions for mathematical statements that Balaguer offers (pp. 89–90):

> In order for it to be the case that '3 is prime' is true, it needs to be the case that (a) there is at least one object that satisfies all of the desiderata for being 3, and (b) all the objects that satisfy all of these desiderata are prime. Or more simply, it needs to be the case that (a) there is at least one standard model of arithmetic, and (b) '3 is prime' is true in all of the standard models of arithmetic.

Note further that these conditions appeal to the property of being 3. Does this property have the number 3 as a constituent? If so, which one? (According to FBP, there is no unique natural number 3.) The conditions also appeal to the property of being prime. But in FBP, aren't there also many different properties of being prime? In the second version of the truth conditions of '3 is prime', an appeal to standard models is made. Does a platonist have to assume the notions of model theory as primitive? Do platonists really have to claim that the truth conditions of number-theoretic statements imply facts about model-theoretic notions? Finally, when Balaguer explains the truth of '3 is prime' in terms of ' "3 is prime" is true *in* all of the standard models of arithmetic', which notion of truth is basic for the platonist?

Similarly, the attempt to formulate fictionalism in a way that doesn't presuppose abstract objects (pp. 12–14) is never quite completed. For example, fictionalists would claim that neither '3 is prime' nor '4 is prime' is true, but claim that '3 is prime' is true in the story of mathematics. But how do we account for the story of mathematics without invoking propositions, contents, sets of sentence types, or other abstract objects? Balaguer argues (p. 14) that this problem is the same as another problem for fictionalism, namely, how to account for the applicability of mathematics. But this argument (which we won't describe here) didn't convince us; it's unclear that one can use Balaguer's subsequent distinction between the 'nominalistic content' and 'platonistic content' to formulate fictionalism so precisely that ' "3 is prime" is true in the story of mathematics' doesn't imply the existence of abstract objects. These questions, and the ones in the previous paragraph, suggest that the formulations of FBP and fictionalism need more work if we are to feel confident about the conclusions of the arguments Balaguer develops concerning these theories.

(2) Crucial to Balaguer's ultimate conclusions is the rejection of confirmational holism. This is important in two related ways. The first is that in order to convince us that FBP is the only viable form of platonism, Balaguer argues that all other platonist positions fall foul of Benacerraf's epistemological challenge, and his case against Quinean platonism depends to a large extent on the rejection of holism.[8] The second way in which the rejection of holism is important to Balaguer's case is in his defense of fictionalism, where he argues that fictionalism can answer the Quine-Putnam indispensability argument by denying confirmational holism.[9] Thus, if Balaguer's arguments against holism do not go through, instead of being left with one viable form of platonism (FBP) and one viable form of antiplatonism (fictionalism), he is left with *two* viable forms of platonism (FBP

[8] This is because Quine claims that we come by mathematical knowledge in exactly the same way as other forms of knowledge—by the empirical confirmation of whole theories in which mathematics plays indispensable roles. Balaguer suggests that this Quinean epistemology does not adequately meet Benacerraf's epistemological challenge because, *contra* holism, only the nominalistic content of scientific theories is confirmed by empirical evidence. (Sober [1993] has argued for a similar conclusion and Balaguer might have cited Sober in further defense of this claim.)

[9] Balaguer believes that mathematical entities can be dismissed as useful fictions in the scientific enterprise, whether they are indispensable or not, largely because they lack causal powers and spatio-temporal location.

and Quinean platonism) and *no* viable form of anti-platonism. Clearly a great deal hangs on his arguments against confirmational holism and yet there are some problems here that need to be addressed.

For instance, consider Balaguer's argument for the claim that empirical science does not confirm the existence of mathematical objects.

> Empirical science *knows*, so to speak, that mathematical objects are causally inert. That is, it does not assign any causal role to any mathematical entity. Thus, it seems that empirical science *predicts* that the behavior of the physical world is not dependent in any way upon the existence of mathematical objects. But this suggests that what empirical science says about the physical world—that is, its complete picture of the physical world—could be true even if there aren't any mathematical objects. (p. 133)

Putting aside worries about whether the abstract-concrete distinction is as sharp as Balaguer supposes,[10] it is not clear that the physical universe cannot depend upon causally inert (or at least causally isolated) entities. After all, physicists posit causally isolated universes (*i.e.*, universes with no causal influence on *this* universe) in order to explain why our universe is fine-tuned for carbon-based life.[11] (Admittedly these universes are not taken to lack causal powers *simpliciter*—they are just taken to lack causal influence on this universe—but it's hard to see why this would be a saliently relevant difference.) It seems, then, that certain features of the physical universe (namely its 'fine-tuning') may be explained by appeal to causally isolated entities (*i.e.*, other universes) and thus, in some sense, the physical universe may indeed be said to depend upon causally isolated entities.[12] Another, related concern with Balaguer's rejection of holism is the nagging doubt that it is just a bit too easy. He claims that 'the nominalistic content of empirical science is all empirical science is really "trying to say" about the world' (p. 141). Indeed, this is the driving force behind his rejection of holism and yet it seems to be little more than an intuition in favor of nominalism. For surely at least part of the business of science is to describe reality. To suppose that reality can be described by the nominalistic content of scientific theories is something akin to begging the question against the platonist. Balaguer might have done a little more to alleviate such doubts.

(3) Although Balaguer does solve the uniqueness problem that affects traditional platonism (as we outlined in Section 1.1), FBP lands him in the middle of a *new* uniqueness problem which is not simply a variant of the problem Benacerraf posed in [1965]. As soon as a platonist postulates a plenitude of mathematical objects, it becomes a question as to how the singular terms of our most fundamental mathematical theories can have

[10] See Resnik [1997], chapter 6 for reasons to suppose that the distinction is not sharp.

[11] We are not endorsing this hypothesis; we are just pointing out that it is an hypothesis that is entertained by physicists, and for all we know it might be true.

[12] See Colyvan [1998] for more on this.

denotations. If all possible sets exist, there will be an ω that exists in virtue of the truth of ZF + CH, an ω that exists in virtue of the truth of ZF + $\neg$CH, an ω that exists in virtue of the truth of ZF + AC, *etc.* So which of these sets does the singular term 'ω' that occurs in ZF denote? Indeed, how can a mathematician working solely in ZF have *de re* beliefs about ω, since FBP rules that there must be massive indeterminacy here? A traditional, non-plenitudinous platonist assumes that there is exactly one true set theory, and so may suppose that 'ω' in ZF has a unique denotation. But this is not an option for Balaguer's full-blooded platonist.

Notice that FBP therefore cannot give the usual 'face-value' interpretation of mathematical claims, since singular terms don't have unique denotations. Balaguer acknowledges this and recovers a kind of face-value interpretation by distinguishing a 'thick' and 'thin' sense in which mathematical theories are about mathematical objects. But the problem goes much deeper and threatens to undermine the entire enterprise, since much of the original motivation for FBP was to account for how our mathematical beliefs (many of which seem clearly to be *de re* beliefs) constitute knowledge. Remember that one of the most serious obstacles to traditional platonism was to give an acceptable account of how we come to believe and know such alleged facts as that 3 is prime, that $\emptyset \in \{\emptyset\}$ and that ω is a limit ordinal. Platonism just is in part the view that these are singular truths of some kind that involve specific abstract objects, but FBP seems to abandon this view.

(4) Balaguer is certainly aware of the problem just outlined and spends much of Chapter 4 attempting to *argue* that mathematical theories do not describe unique collections of abstract objects, that the terms of our mathematical theories do not have unique denotations, and that we do not therefore have *de re* mathematical beliefs. Unfortunately, these arguments do not consider either the literature that addresses the original uniqueness problem posed in Benacerraf [1965] or the literature describing a version of platonism that is both 'full-blooded' *and* immune to both uniqueness problems. In [1987], George Boolos introduced on Frege's behalf a second instantiation relation, $F\eta x$ ('F is in x'), and formulated 'Frege Arithmetic' as a way to conceive of numbers as metaphysically-distinguished abstract objects.[13] Had Balaguer considered the implications of Frege Arithmetic, he might have rejected the premise that there is no unique sequence of abstract objects identifiable as the natural numbers instead of rejecting the premise that traditional platonism entails that there is such a sequence (these were two of the premises in the argument in terms of which Balaguer

[13] Frege Arithmetic is simply second-order logic augmented with the following principle:

$$\text{Numbers: } \forall G \exists! x \forall F (F\eta x \equiv F \approx G),$$

where '$F \approx G$' asserts that F and G are in one-to-one correspondence and is given the usual second-order definition.

formulated 'the uniqueness problem', pp. 76–77).

Given the present context, there are reasons not to consider Frege Arithmetic, namely the fact that it is not a full-blooded platonism and it doesn't come with a naturalized epistemology. But then, one of the present authors has developed a theory of abstract objects which is based on a plenitude principle that is immune to both uniqueness problems and which has been shown (in a co-authored paper) to have a naturalized epistemology.[14] This theory also uses a second mode of predication (x encodes F) and an identity principle that individuates *abstracta* by the properties they encode rather than by the properties they exemplify. Mathematical objects are identified as abstract objects that encode *only* the mathematical properties attributed to them in their respective mathematical theories.[15] In the context of such a metaphysics, mathematical theories *do* pick out unique collections of abstract objects—each mathematical theory T describes the unique collection of abstract objects that encode exactly the properties attributed to the mathematical objects in T.[16] At the very least, this philosophy of mathematics reveals that Balaguer's arguments for the indeterminacy in our mathematical beliefs and for the rejection of *de re* mathematical beliefs (which come to a head in the penultimate paragraph of Section 4.2) are inconclusive at best.[17]

(5) The most fascinating part of Balaguer's book is the last chapter, in which he: (a) summarizes the numerous points on which full-blooded platonists and fictionalists agree, (b) describes the one focal point on which they disagree, namely, on (the consequences of) the claim that abstract objects exist, (c) argues that the truth conditions for 'abstract objects exist' are not clear, and (d) then concludes that there is no fact of the matter as to whether abstract objects exist and so no fact of the matter as to whether FBP or fictionalism is true. This is intriguing anti-metaphysical work, but in defense of metaphysics, there is an hypothesis concerning the above (in the literature) which Balaguer never quite explicitly formulates but which seems consistent with his view. This hypothesis (a) explains the equivalence of full-blooded platonism and fictionalism on the eight points

[14] See Zalta [1983] (Chapter VI) and Linsky and Zalta [1995].

[15] As such, they are 'incomplete' with respect to the properties they encode, though complete with respect to the properties they exemplify, such as being non-red, being non-round, being thought about by x, *etc.*

[16] Since every theory is about a unique collection of abstract objects, it follows *a fortiori* that Peano Number Theory is about a unique collection of objects, *i.e.*, the ones that encode just the properties attributed to the numbers in Peano Number Theory. So the old uniqueness problem is solved as well.

[17] This philosophy of mathematics also has consequences for his objection to the structuralist idea that, in some sense, mathematical objects have only mathematical properties. See the discussion of Parsons's suggested formulation of structuralism on pp. 9–10 and the discussion of structuralism on pp. 80–84.

Balaguer describes, (b) offers an explanation as to how two theories that agree on so much can yet differ on the claim that abstract objects exist, (c) offers precise truth conditions for the claim that abstract objects exist, and (d) yields an explanation as to why we may not be able to determine a fact of the matter as to whether abstract objects exist. The hypothesis is that there is a single formal (plenitude) theory of abstract objects which has *two* interpretations, one platonist and one fictionalist, and which has the following features.[18] When natural-language claims such as '3 is prime' are represented in the formalism, they have one reading on which they turn out true and a distinct reading on which they turn out false. The formal theory uses a quantifier ('$\exists$') and a predicate ('A') to assert '$\exists x Ax$', but the formalism itself doesn't determine whether: (i) $\exists$ is to be read as existentially loaded (as Quine suggests in [1948]) and A is to be read 'abstract', or (ii) $\exists$ is to be read as existentially *un*loaded (as T. Parsons suggests in [1980] and Azzouni mentions in [1998]) and A is to be read 'nonexistent fiction'. Thus, the plenitude principle (expressed in terms of the quantifier and predicate) would describe either a plenitude of existing abstract objects or a plenitude of nonexisting fictions, depending on the interpretation (i) or (ii).

Such an hypothesis would allow us to reason as follows. (a) The numerous points of agreement between full-blooded platonism and fictionalism are explained by the fact that they use the same formalism to approach substantive issues. (b) That they could agree so substantially and yet differ on (the consequences of) the *single* issue concerning the existence of *abstracta* is explained by the fact that they are two incompatible interpretations of the same formalism.[19] (c) The precise *platonist* truth conditions which Balaguer demands for the claim 'abstract objects exist' will now issue not only from the definition of the predicate 'A' in terms of primitive (logical and nonlogical) notions that Balaguer uses and accepts, but also from the fact that *the axioms* governing 'A' tell us exactly what the world would have to be like for *abstracta* to exist.[20] (d) Finally, if it turns out that

[18] See Zalta [1983], both p. 51 and (its application to) Chapter VI, Section 4. Zalta's theory is just one way of working out the hypothesis in question; other formal theories of abstract objects with these features are possible.

[19] When the platonist accepts that abstract objects exist and concludes that our mathematical theories are true, she can represent '3 is prime' as a truth, since the formalism offers such a reading. Similarly, when the fictionalist claims that mathematical objects are fictions, she can represent '3 is prime' as a falsehood, since the formalism offers such a reading as well.

[20] To be precise, let 'C' express spatiotemporality and let 'Ax' ('x is abstract') be defined either as '$\neg Cx$' (this is the definition that Balaguer take to be unclear) or as '$\neg \Diamond Cx$' (where $\neg$ is classical and $\Diamond$ is axiomatized by classical S5 modal logic). Balaguer uses and seems to understand all of these notions, and so the platonistic truth conditions for the claim Ax are: x is not (the kind of thing that could be) spatiotemporal. Consequently, the assertion that abstract objects exist (*i.e.*, the platonistic interpretation of '$\exists x Ax$')

our actual linguistic practices are indeterminate and can be systematized in one of two equally precise ways, each of which uses the formalism under one of its interpretations, we have an explanation of why the question of whether abstract objects exist can't be settled (though, this is not quite to say that there is no fact of the matter).

Even though Balaguer doesn't consider this hypothesis as a way of tying up the four strands (a)–(d) in the final chapter, his book is significant for identifying a deep connection between two apparently opposed philosophies of mathematics. We are convinced that Balaguer is onto something, even if the arguments that get us there are not always airtight. It is a remarkable accomplishment that he didn't lose sight of the overall forest for the trees.[21]

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has well defined truth conditions, at least relative to our understanding of negation, quantification, spatiotemporality, and S5 modality. Now each axiom in the formal theory (under this interpretation) tells us a little more about what the world would be like for abstract objects to exist.

[21] We are indebted to Mark Balaguer and J. C. Beall for reading and commenting on an earlier draft of this review.

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