Noûs

NOÛS 45:2 (2011) 345–374

An Inferential Conception of the Application of Mathematics

OTÁVIO BUENO†
University of Miami

MARK COLYVAN†
University of Sydney

Abstract

A number of people have recently argued for a structural approach to accounting for the applications of mathematics. Such an approach has been called "the mapping account". According to this view, the applicability of mathematics is fully accounted for by appreciating the relevant structural similarities between the empirical system under study and the mathematics used in the investigation of that system. This account of applications requires the truth of applied mathematical assertions, but it does not require the existence of mathematical objects. In this paper, we discuss the shortcomings of this account, and show how these shortcomings can be overcome by a broader view of the application of mathematics: the inferential conception.

1. Introduction

There is a fairly widespread view that mathematics is applicable to the world simply because some portion of the mathematical universe shares some structural similarity with a portion of the physical realm. So, for example, the real numbers are (arguably) isomorphic to a single dimension of space, and this is why real analysis can be so effectively used in kinematics. This structural similarity may, in the best cases, be an isomorphism, though in general it will consist simply in a mapping of some kind or other. Of course, there will be different mappings in different applications. According to this view of mathematical applications—the "mapping account", as Christopher Pincock (2004a) has called it—the existence of an appropriate mapping from a mathematical structure to a physical structure is sufficient to fully explain the particular application of the mathematical structure in question.

In this paper, we will show that this account of mathematics is correct as far as it goes, but that it does not go far enough. We will argue that quite a bit more is required for an adequate account of applied mathematics, and we will advance an account of applied mathematics that accommodates central features of the application process, including the structural elements correctly described by the mapping account. We call this view *the inferential conception of applied mathematics*. We also argue that this view is neutral on the realism/anti-realism issue in the philosophy of mathematics, and so can be adopted by both sides of the realism/anti-realism divide.

2. Mappings and Structural Similarities

Why is it that mathematics is so useful in empirical science? One answer is simply that mathematics is a rich source of structures and therein lies its utility. Some mathematical structure is either designed to, or otherwise found to, accurately capture the important structural relations of an empirical set up, and we can thus read off important facts about the empirical set up from the mathematics.[1]

On this view, the usefulness of mathematics is much like the usefulness of a city street map. Once the structural similarity between the map and the city's streets is established, the map can stand in for the world. Moreover, various hard-to-determine facts about the city can be read off rather easily from the map. All that is required for a map to be useful is that it faithfully represents some aspects of the city in question. It is important to note that the map need not faithfully preserve all structural relations. For example, the street map must preserve directional relations and is usually required to preserve (scaled) distance relations. We have conventions in place to help here: if point A is north of point B in the city, then the representation of A needs to be above the representation of B on the map. And distance relations are not literally preserved in the map—otherwise the map would be useless. Instead, we invoke a transformation according to which a kilometer in the city is represented in the map by a smaller length—a centimeter, say. But we should also note that certain structural relations of the city are *not* represented in the street map. The gradient of the streets is typically not represented. Indeed, much of the utility of the map derives from the fact that it is quite deliberately less complex than the city it represents. In other words, the city has more structure than is represented in the map.[2] It is also worth mentioning that for a street map to be accurate, there must be a structure-preserving mapping (in the mathematical sense) between the city and its map (in the usual, non-technical sense). This should all be familiar enough.

According to the mapping account of mathematical applications, the explanation of the utility of mathematics is no different from explaining the utility of street maps. The idea is that there is some structure-preserving mapping between the world and the mathematical structure in question, and that is pretty much the end of it.[3] Well, not quite. We need to say a bit more about the kind of mapping (i.e. isomorphism, homomorphism and so on), but that is not thought to be too problematic. Indeed, one defender of this view, Chris Pincock (2004a), feels no need to say anything at all about the kind of mapping between the mathematics and the empirical set up. This is presumably because either the kind of mapping is of no great significance or, more plausibly, the kind of mapping will depend on the application in question.[4]

There is also the issue of what we mean by 'a structure' and, in particular, by 'the structure of the world'. A structure is usually taken to be a set of objects (or nodes or positions) and a set of relations on these (Resnik 1997; Shapiro 1997). We will adopt this standard account here. But with this definition of structure in place, a difficulty arises when we try to talk of the structure of the world. Put simply, the world does not come equipped with a set of objects (or nodes or positions) and sets of relations on those. These are either constructs of our theories of the world or identified by our theories of the world. Even if there is some privileged way of carving up the world into objects and relations (and, of course, it is extremely controversial that the world co-operates in this way, providing natural joints, as it were), such a carving, it would seem, is delivered by our theories, not by the world itself. What we require for the mapping account to get started is something like a pre-theoretic structure of the world (or at least a pre-modeling structure of the world). This is clearly a problem for the mapping account.[5] In the street-map example it seems natural enough to divide the world up into streets, rivers, coastlines and the like, but in general this will not be the case. When there is a natural candidate pre-theoretic structure, the mapping account can employ this structure. When there is no such structure, we might impose some suitable structure or other and let the resulting mathematical model help us to fine tune or revise the starting structure. Either way, the mapping account does require having what we shall call an *assumed structure* in order to get started. There is no avoiding such an assumption.

In the next section, we argue that defenders of the mapping account of mathematical applications need to say a great deal more about the kind of mappings used. In what follows, we will primarily focus on Pincock's (2004a) detailed presentation of the mapping account, though it is worth noting that others clearly endorse something like this view (e.g., Baker 2003, Balaguer 1998, p. 144, and Leng 2002), and we will use the term "mapping account" to refer to what we take to be the core of all such views. While the difficulties we raise for the mapping account may well be able to be addressed by that view, our primary purpose is to outline the inferential view of applied mathematics. We will thus take what we see as shortcomings of the mapping account as a motivation for the view we will present in sections 5 and 6. But we also hope to preserve what is correct about the mapping account.

3. Shortcomings of Pure Structuralism

There is clearly something right about the mapping account. Mathematics *is* a rich source of structures and when a mathematical theory finds applications in empirical science, it is clear that the mathematics captures certain important structural relations of the system in question. We are in full agreement with the defenders of the mapping account thus far. The disagreement is over whether this is all that needs to be said. After all, it might be argued that the mapping account does not need to say anything about the kind of mapping that relates the target system and the corresponding model.

In this section, we will argue that the mapping account is right as far as it goes. But, as it stands, it is incomplete in a couple of ways. First, until the central notion of *mapping* is clarified, the account is little more than a gesture at an account. Second, certain aspects of the applications of mathematics do not sit well with the mapping account, or at least the account does not address some important issues arising in the application of mathematics. We discuss each of these complaints in turn.

Pincock (2004a) quite rightly suggests that the mapping between the mathematical structure and the world need not be anything so tight as an isomorphism. After all, in most cases it is patently clear that there is more structure in either the world or the mathematics. But for the mathematics to be useful, it must faithfully represent structural relations in the world. So, it would seem that the mapping cannot be just *any* mapping. Let's introduce a little formalism before we explore this issue further.

Let W be the empirical situation to be represented (or "world"), let M be the mathematical structure to be used in the model of W, and let φ: $W \to M$ be the mapping in question. There are a few structure-preserving maps besides isomorphisms to consider: homomorphisms, epimorphisms, and monomorphisms.[6] It would seem that the mapping employed will depend on the richness of the two structures in question, W and M. Suppose that the world has more structure than the mathematics, in the sense that there are either more objects or more structural relations between them in the world than can be represented in the mathematical structure.[7] In this case, we have three options: φ can fail to be a homomorphism at all, it can be a homomorphism, or can be surjective and thus be an epimorphism (i.e. it can't be a monomorphism)—in all three cases, some of the elements or relations in W are not faithfully represented in M. However, if the mathematics has more structure, then we now have a fourth option in addition to the two previous ones: φ can be a monomorphism. (For practical purposes, however, when we mathematically represent certain phenomena, we would rather have mappings that let us preserve as much structure as possible given the problem at hand, in which case the monomorphism option is the most significant when the mathematics has more structure than the world.) But now the problem arises that in order for the mathematics to be useful, we will need to be able to move freely into and out of the mathematics, just as we can move freely between our street directory and the city.[8] We thus need φ to be invertible. This means φ needs to be a monomorphism. Furthermore, this, in effect, rules out W having more structure than M. But often the world *is* more complex than the mathematics used to model it. (Consider, for example, modeling time with a 12-hour clock—essentially arithmetic mod 12. As it stands such a clock does not allow one to make sense of 2 am on different days, for instance.) We'll return to these issues shortly, but for the moment, we simply note that there are some problems here for the mapping account. As currently formulated, the account is rather limited in the mappings it has at its disposal. Excluding isomorphisms, which as noted are often too tight, only monomorphisms seem a viable option.

So on the view under consideration, every application of mathematics to empirical science must involve a monomorphism φ from W to M. But because φ is not necessarily surjective, we may have it that the range of φ is a proper subset of the codomain M. That is, there may be elements of M that have no pre-image under φ. But how do we then interpret such elements in the mathematical structure? At first blush, this might not seem like a problem. The mathematical domain is richer than the empirical domain. So, there will be redundant structure in M, and this structure will play no real role. But recall that the purpose of using the mathematics is to model the empirical situation and then presumably solve problems in the mathematics that correspond to empirical problems. What if the solutions to the mathematical problem have no empirical counterpart? For example, consider a typical projectile problem in which one needs to calculate where a projectile (of known initial velocity and position whose only acceleration is due to gravity) will land. Of course, the displacement function for such a projectile is a quadratic with two real solutions—only one of which is physically significant.[9] The problem is how do we know which parts of the mathematical structure represent and which parts do not? The mapping account is silent on this issue.

In the simple projectile example above, it is easy enough to tell that there should be a unique solution to the problem. Moreover, it is (almost) always clear which solution is physically significant and which is not. Both are provided by reliable physical intuitions—projectiles do not land in more than one place and they typically land forward of their launch site (in the direction of the horizontal component of their initial velocity). The issue is that such crucial information required to solve this physical problem *is not part of the mapping between mathematical structure and physical structure*. In short, the mapping account of mathematical applications is incomplete.

We use a simple example like this quite deliberately. The phenomenon we are drawing attention to is present in even the most basic of applied mathematics problems. But as the mathematical applications get more complex, not only does the phenomenon persist, it is often more difficult to tell whether the mathematical solutions are physically significant or not. Consider, for example, the generalized Dirichlet problem of finding a harmonic function on the interior of a closed bounded region of the complex plain such that the function agrees everywhere with a function defined on the boundary of the region in question (Ahlfors 1979, pp. 245–251). This problem crops up in many physical situations such as potential theory. But the problem does not, in general, have a solution in cases where the boundary function is discontinuous. How are we to interpret such failures to find a solution? Are such discontinuous boundary functions physically significant? It depends on the physical application in question, and even then, the answer is not easy. Furthermore, in some cases, what appear to be non-physically significant solutions turn out to be physically significant. Consider, for example, time dilation in Lorentz's (1904) pre-special relativity theory of length contraction. Lorentz seemed to take the time-dilation effect as a mere artifact of the mathematics—that is, not physically significant—presumably because of rather natural (pre-relativistic) intuitions about the nature of time. As we now know, Lorentz was wrong about this feature of the mathematics being an artifact. But here we have an interesting feature of applied mathematics that needs explaining: very often the mathematics employed either captures more physical structure than was intended, or solutions that, from physical considerations alone, appear not to be physically significant yet turn out to be physically significant. As Heinrich Hertz once suggested, mathematics, it seems, is wiser than we are.

This is one aspect of what has become known as "the unreasonable effectiveness of mathematics" (Wigner, 1960), and has been made much of by Mark Steiner (1998) (see also Colyvan 2001b). Here we are not claiming that this problem has no solution, just that, as it stands, the mapping account of applied mathematics provides no solution. At the very least, the mapping account is incomplete as a philosophical account of applied mathematics. Moreover, the incompleteness of the mapping account is seen clearly as a result of problems relating to the specifications of the mappings in question.

It might be argued that we are setting the bar too high here, in that we are expecting a philosophical account of applied mathematics to provide the details of which parts of mathematical models refer and which do not *in any given application*. Surely, the objection continues, we would expect such issues to be sorted out by context on a case-by-case basis. Our response is to point out that we are not setting the bar so high as this. We are simply demanding that a philosophical account of applied mathematics *says something about this issue*. It may well be that a thorough treatment of the issue will rely on the context of the individual cases, and that there will be no fully general story about which parts of a mathematical structure represents and which do not. But to remain completely silent on this issue is to fail to address one of the more interesting and important issues in the philosophy of applied mathematics. Any such account of applied mathematics is, at best, incomplete. Silence here amounts to simply acknowledging that sometimes the mathematics refers to the world and sometimes it doesn't, and there is nothing more to be said about it. But that strikes us as giving up too easily. Without some story about this issue that reveals how the resulting mathematical models are useful, a serious philosophical issue is left hanging. We take it as a desideratum of an adequate philosophical theory in any domain that it does not leave matters hanging in this way.

Another related problem with the mapping account occurs in cases when there is known mismatch between the empirical structure and the mathematical structure. Consider, for example, the various idealizations made in fluid mechanics (e.g., fluids are incompressible) or the various assumptions introduced in economics (e.g., agents are ideally rational). Here there seems to be no mapping between the empirical structures in question and the mathematical structures. If anything, there would appear to be a mapping between the mathematical structure and some possible, but non-actual empirical structure. Such problems concerning mathematical models are well known and have been brought to prominence in philosophy of science by Nancy Cartwright (1983). Again, it might be thought that the various mismatches between mathematics and the system under study are a problem for most accounts of applied mathematics, so it might seem a little unfair to direct this problem at the mapping account. But plausibly the mapping account is supposed to be a complete account of applied mathematics. Until it can provide a satisfactory account of such idealizations, the account is at best incomplete.[10]

Finally, we turn to the matter of explanation. Implicit in the mapping account is the assumption that mathematics is no more than a convenient representational system. The mathematics represents various features of the physical world via the mappings. But there would seem to be cases where mathematics plays other roles. First, and least controversially, mathematics is able to unify various disparate phenomena (Colyvan 2001a, 2002). But this kind of role may not be too problematic for the mapping account (see Pincock 2007).

If mathematics is genuinely explanatory, however, this will present a serious problem for the mapping account. The problem is simply that it is hard to see how a mere representational system can provide explanations and yet that is the only role mathematics is allowed to play in the mapping account.[11] Consider once again our map of a city. Certain facts about the city will be more obvious in the street map—indeed, that's the purpose of a street map—but it would be very odd to think of the map as providing an *explanation* of any facts about the city. At least, it would be odd to think of the street map offering an explanation that wasn't just standing proxy for another physically-significant explanation.[12] But it seems to at least one of the present authors that, in many circumstances, mathematics may yield explanations of various physical phenomena. Consider the algebraic explanation of why, with compass and straight edge, one cannot square the circle:

because pi is transcendental. Or consider the (standard Lotka-Volterra) explanation of why all populations whose abundance exhibits cycles must be part of a predator–prey pair: because there are no periodic solutions to first-order differential equations and coupled first-order differential equations are equivalent to a second-order differential equation (and the latter do allow periodic solutions).[13]

We won't pursue such examples further here. We admit that the case for mathematical explanations is controversial, but there is at least a *prima facie* case for such explanations. And even this is enough for the burden of proof to lie with the mapping account of mathematical applications. Defenders of this view must either show how the mapping account can accommodate mathematical explanations, or they must show that there are no such explanations. Either way, considerable work is required on the mapping account before it can be considered an adequate account of applied mathematics.[14]

4. An Alternative View: An Inferential Conception of Applied Mathematics

We now offer an alternative conception of the application of mathematics, indicating in which ways it moves beyond the mapping account. In fact, the proposal is an extension of the latter account, in that it agrees that mappings of a variety of sorts are crucial to applied mathematics. Unlike the mapping account, however, the proposal advanced here is not purely structural, since it makes room for additional pragmatic and context-dependent features in the process of applying mathematics.

The crucial feature of the proposal is that the fundamental role of applied mathematics is inferential: by embedding certain features of the empirical world into a mathematical structure, it is possible to obtain inferences that would otherwise be extraordinarily hard (if not impossible) to obtain. Now, this doesn't mean that applied mathematics doesn't have other roles. We mentioned already several such roles: from unifying disparate scientific theories through helping to make novel predictions (from suitably interpreted mathematical structures) to providing explanations of empirical phenomena (again from certain interpretations of the mathematical formalism).

All of these roles, however, are ultimately tied to the ability to establish *inferential relations* between empirical phenomena and mathematical structures, or among mathematical structures themselves.[15] For example, when disparate scientific theories are unified, one establishes inferential relations between such theories, showing, for example, how one can derive the results of one of the theories from the other—this is, arguably, one of the points of unifying the theories in the first place. Similarly, in the case of novel predictions, by invoking suitable empirical interpretations of mathematical theories, scientists can draw inferences about the empirical world—leading, in certain cases, to novel predictions—that the original scientific theory wasn't constructed to make. Whether novel predictions are used to support a realist reading of the theories involved or whether anti-realist can also make sense of them, the important point for us here is that both parties should agree about the importance of *inference* in the production of novel predictions. Finally, in the case of mathematical explanations, inferences from (suitable interpretations of) the mathematical formalism to the empirical world are established, and in terms of these inferences, the explanations are formulated. These cases illustrate the crucial *inferential role* that applied mathematics can play.[16]

To accommodate this inferential role, it's indeed crucial to establish certain mappings between the empirical set up and appropriate mathematical structures. The mapping account is correct in highlighting the centrality of these mappings. But we have to be explicit about (i) the kind of mappings that should be invoked in each context, and (ii) how to accommodate the inferential role of applied mathematics. In order to do that, we provide the following three-step scheme (see Figure 1):

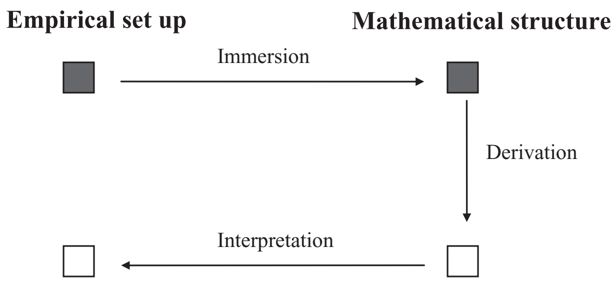

Figure 1. The Inferential Conception of Applied Mathematics.

(a) The first step consists in establishing a mapping from the empirical set up to a convenient mathematical structure.[17] We call this step *immersion*. The point of immersion is to relate the relevant aspects of the empirical situation with the appropriate mathematical context. The empirical situation is taken very broadly, and it includes the whole spectrum of contexts to which mathematics is applied. (In the limit, this includes mathematical contexts as well, such as when mathematicians apply set theory to arithmetic in order to obtain new results about the latter.) As we will see, several mappings can do the job here, and the choice of mapping is a contextual matter, largely dependent on the particular details of the application.
(b) The second step consists in drawing consequences from the mathematical formalism, using the mathematical structure obtained in the immersion step. We call this step *derivation*. This is, of course, the key point of the application process, where consequences from the mathematical formalism are generated.
(c) Finally, we interpret the mathematical consequences that were obtained in the derivation step in terms of the initial empirical set up. We call this step

> *interpretation*. To establish an interpretation, a mapping from the mathematical structure to the initial empirical set up is needed. This mapping need not be simply the inverse of the mapping used in the immersion step—although, in some instances, this may well be the case. But, in some contexts, we may have a different mapping from the one that was used in the immersion step. As long as the mappings in question are defined for suitable domains, no problems need emerge.[18]

It is worth drawing attention to the differences between our approach and the mapping account. For a start (and this is non-trivial), our proposal is clearly laid out in the above three steps. The first two steps, plausibly, bear some similarity to aspects of the mapping account, but the latter account has not been articulated in such clear and explicit terms. So, it is difficult to compare the two in any detail. There is, however, one major and obvious difference, and that is step three. There is nothing that resembles this step in the mapping account.

We also note here that the above sharp distinction between the empirical set up and the mathematical structure does not suggest that there need be a mathematics-free description of the empirical set up. Very often the only description of the set up available will invoke a great deal of mathematics. Thus, it will be hard to even talk about the empirical setup in question without leaning heavily on the mathematical structure, prior to the immersion step. The empirical setup is the relevant bits of the empirical world, not a mathematics-free description of it.

So far, we've accounted for the most straightforward cases of applying mathematics. We can also have embedded cases of applied mathematical structures. In these cases, after the initial immersion step, and before the derivation and interpretation steps, we map the resulting mathematical structure into another mathematical structure, and then apply the two other steps. That is, we derive consequences from the new mathematical structure, and interpret the results that are obtained back into either the initial mathematical structure and then back to the empirical setting, or directly back to the empirical set up. Once again, different mappings are crucial in each case.

The description above is idealized in two respects, since the immersion and the interpretation steps aren't as clean as is suggested. (i) The mathematical formalism often comes accompanied by certain physical "interpretations", and (ii) the description of the empirical set up is often made already in mathematical terms. We are not assuming, in the immersion and the interpretation steps, that the empirical set up and the mathematical structures are completely distinguished components. The crucial point is that however the empirical set up and the mathematical structures are formulated, the application process involves establishing mappings between them. And this is the point of the immersion and the interpretation steps.

It is worth highlighting here that there is considerable choice about the mappings used in both the immersion and interpretation stages. In both cases the decision about the choice of mappings will be a matter of context, and pragmatic considerations come into play. Take the immersion stage first. If, for example, we wish to determine the combined mass of a number of objects, we should use an interpretation mapping that assigns masses—not space-time locations, not lengths, or anything else—to each object. Similarly, at the interpretation stage we need to make a decision about how to interpret the sum obtained, because, in the uninterpreted mathematics, the sum is just a real number. Nothing forces the interpretation of this number to be the combined mass of the objects in question. Of course, in an example like this, the choices in each direction are clear. In other cases, there may be more than one way to get the desired result.[19] And, as we've already argued, the choice of interpretation will not, in general, be the inverse of the immersion map. We will return to the issue of deciding on the suitable immersion and interpretation mappings in section 5.4.

It might be thought that the only relevant structure is causal structure, so the job of mathematics, on both the mapping account and the inferential account, is to model the causal relations in the physical set up in question. Often this will be the case, but it is important to note that the inferential account (and perhaps the mapping account as well) is able to accommodate structure other than causal structure. For example, Robert Batterman (2002, p. 13) argues convincingly that "science requires methods that eliminate both detail and, in some sense, precision". In particular, aspects of causal detail need to be eliminated for some explanatory purposes. An example of Batterman's will help illustrate the point. Consider a stiff ribbon of steel—an Euler strut—mounted vertically on a hard surface and loaded with weights from above. Eventually the load will be sufficient for the strut to bend to either the left or the right. Whether it bends one way or the other will depend on micro-level causal factors—the trajectories of colliding air molecules, micro-structural asymmetries in the strut, and so on. But other, more universal questions, about the behavior of other struts made from the same and from different materials, will require abstracting away from such detail. This more abstract level of description and explanation involves what Batterman (2002, p. 13) calls *asymptotic reasoning* and is characterized by the move away from particular causal detail.

This form of reasoning is very important in science, and one that any adequate account of applied mathematics must be able to accommodate. The inferential account, in so far as it is not tied to causal structure, is well placed to deal with this kind of inference and the examples Batterman examines. To return to the strut case, the choice of maps at both the immersion and interpretation stages will depend on the questions that need to be answered. This is where context comes into play. If we are interested in the general behavior of all such struts, no matter what rigid metal they are made from, then we must abstract away from the particular causal detail of the case at hand. The appropriate immersion map, in such a case, will yield Euler's formula:

$$P_c = \pi^2 EI/L^2,$$

where P_c is the critical buckling load for the strut, E is Young's modulus characteristic of the material, I is the second moment of the strut's cross-sectional area, and L is the length of the strut. Notice that a great deal of the causal detail is left out of this model (there is, for example, no mention of what the surrounding air molecules are doing), and what causal information that is included is presented rather imprecisely (Young's modulus summarizes a great deal of information about the micro-causal structure of the strut). If, on the other hand, we are interested in why the strut buckles to, say, the left, then we will need to represent micro-causal structure such as trajectories of nearby air molecules and so on. The choice of immersion mapping will depend on the questions to be answered and the ultimate purpose of the mathematical model.[20]

5. The Inferential Conception At Work

We will now argue that the inferential conception is not open to the difficulties that we raised above for the mapping account. Along the way, we will also provide examples that illustrate how the inferential conception works.

5.1. Connecting mathematical and empirical structures

As we saw, the mapping account is incomplete in its treatment of applied mathematics. After all, the account doesn't seem to have the resources to accommodate the fact that mathematical theories often have more structure than the empirical set up, and some of that additional structure (suitably interpreted) has empirical implications.

From the point of view of the inferential conception, there is no difficulty here. As we saw, on this conception, the major role of applied mathematics consists in drawing inferences about the relevant domain of application. This domain is typically an empirical set up, but it can also be another mathematical domain, such as in the case of the application of mathematics within mathematics itself. And to carry out these inferences, the richness of structure in each step of the application process is completely dependent on the context. Typically, there will be more structure at the mathematical level than at the empirical one, at least if we consider the cardinality of the domains involved. But this need not be always so. The crucial point is that the inferential conception has the resources to accommodate all such cases, independently of the complexities of the structures involved.

The inferential conception offers a framework in terms of which we can conceptualize two central issues in the application of mathematics: (a) the issue of selecting the appropriate mathematical structures to represent the empirical set up, and (b) the issue of assessing the adequacy of these structures as representational and perhaps explanatory devices. As for (a), the selection process typically emerges from going back and forth between the immersion and the interpretation steps. (We need not think of the immersion step as being logically prior to the interpretation step.) For example, we may select a mathematical structure that is computationally tractable, but which produces empirically inadequate results. Or we may select a structure that, despite its computational complexity, offers interpretations that are empirically well supported. The choice here will emerge from a careful consideration of the benefits and costs of each option. In making this choice, we are also assessing the adequacy of the mathematical structures under consideration, which addresses the issue in (b).

Since the mapping account doesn't provide such a clear framework to conceptualize these issues, the inferential conception seems to be in a better position to help us understand the connections between the empirical set up and the mathematical structures. The details will emerge below when we consider various case studies.

Finally, the inferential conception is well placed to help provide the crucial assumed structure. Recall that the assumed structure is the structure the modeling exercise assumes to be present in the world (or empirical set up, as we were calling it here). Apart from cases where the assumed structure is obvious (as in the street-map example discussed in section 2), we will need to impose some structure on the world in order to begin the modeling exercise. Earlier we suggested that the mapping account might treat this initial assumed structure as defeasible and let the resulting mathematical model help inform refinements or revisions to the initial assumed structure. The inferential account has the resources to make revisions midstream and does not require starting from scratch each time a more fruitful assumed structure is conceived. This is achieved by employing the composite mappings to move from the mathematized initial assumed structure to another mathematical structure, where the latter may be thought to correspond to a new (revised) assumed structure. There is no need to revise formally the initial assumed structure because the interpretation step of the process will deliver the final structure of the empirical set up—one informed by the modeling exercise and one that may well be quite different from the initial assumed structure. The mapping account may be able to solve the assumed structure problem (as we have already suggested), but via more ad hoc and trial-and-error methods. The increased flexibility of the inferential account allows us to approach revisions to the assumed structure in a smoother more systematic way.[21]

5.2. Accommodating idealizations: a framework

As we also saw, it's unclear how the mapping account could accommodate idealizations and other known mismatches between the mathematical structures and the empirical settings. One way of addressing this problem from the perspective of the inferential conception is, first, by agreeing that, in cases involving idealizations, there's no full mapping between the empirical set up and the mathematical structures. After all, nothing in the empirical world literally and completely corresponds to what is being presupposed in cases involving idealizations. However, although there are no *full* mappings between the empirical world and the mathematical structures, there are *partial* mappings between these empirical and mathematical structures. Certain features of the empirical set up—although not all—can be mapped into appropriate mathematical structures. In contexts where idealizations are employed, the existence of a partial mapping between the empirical and the mathematical structures explains in which respects the idealizations work. The latter capture certain elements of the actual world, but not all of them.

But how can these partial mappings be formulated? A formal framework that represents these mappings very naturally is provided by the partial structures approach (see da Costa and French 2003; Bueno, French, and Ladyman 2002; and French and Ladyman 1998). The idea is that if there's no complete information about a certain domain of investigation, we can represent formally the partiality of that information and the structural relations between the various components involved in terms of the notions of partial structure and partial relation. A partial structure is an ordered par $\langle D, R_i \rangle_{i \in I}$, where D is a non-empty set and R_i, $i \in I$, is a family of partial relations (and I is an index set). A partial relation R_i, $i \in I$, over D is a relation which is not necessarily defined for all n-tuples of elements of D. The partiality of these relations can be interpreted in two ways: (i) It can be interpreted *ontologically*, as representing the incompleteness or partialness of the relations linking the elements of D, or (ii) it can be interpreted *epistemically*, as representing the incompleteness or partialness of our information about the actual relations linking the elements of D. (The formalism suggested here is neutral on this issue, and it can be interpreted in either way.) More formally, each partial relation R can be viewed as an ordered triple $\langle R_1, R_2, R_3 \rangle$, where R_1, R_2, and R_3 are mutually disjoint sets, with $R_1 \cup R_2 \cup R_3 = D^n$, and such that: R_1 is the set of n-tuples that (we know) belong to R; R_2 is the set of n-tuples that (we know) do not belong to R; and R_3 is the set of n-tuples for which it is not defined whether they belong or not to R. (Note that when R_3 is empty, R is a normal n-place relation that can be identified with R_1.)

In terms of partial structures, it's possible to define various forms of partial mappings between these structures, such as partial isomorphism and partial homomorphism. These partial mappings straightforwardly extend the usual notions of isomorphism and homomorphism to partial contexts, and they can be defined as follows. Let $S = \langle D, R_i \rangle_{i \in I}$ and $S' = \langle D', R_i' \rangle_{i \in I}$ be partial structures, where R_i and R_i' are (for simplicity) binary partial relations. We say that a *partial* function f: $D \rightarrow D'$ is a *partial isomorphism* between S and S' if (i) f is bijective, and (ii) for every x and $y \in D$, $R_1xy \leftrightarrow R_1'f(x)f(y)$ and $R_2xy \leftrightarrow R_2'f(x)f(y)$. So, when R_3 and R_3' are empty (that is, when we are considering total structures), we have the standard notion of isomorphism. Moreover, we say that a *partial* function f: $D \rightarrow D'$ is a *partial homomorphism* from S to S' if for every x and every y in

D, $R_1xy \rightarrow R_1'f(x)f(y)$ and $R_2xy \rightarrow R_2'f(x)f(y)$. Again, if R_3 and R_3' are empty, we obtain the standard notion of homomorphism as a particular case.[22]

Since in idealized contexts there's no full mapping between the relevant empirical and mathematical structures, it's crucial that the mappings in question be partial. However, even in idealized contexts, certain aspects of the actual empirical situation can still be mapped to relevant features of the mathematical structures invoked. The R_1 and R_2 components of the partial relations in the empirical set up are mapped via the relevant partial isomorphism or partial homomorphism into the corresponding partial relations in the mathematical structure. However, the R_3 components are left open. These components correspond to the features of the idealization that bring mismatches between the actual empirical world and the mathematical model.

5.3. Accommodating idealizations: an example from economics

To illustrate how the framework above functions, let's consider an example from economic theory. In neo-classical economics, agents are modeled as being perfectly rational (say, as maximizing their utility or expected utility functions). But this is not how agents actually behave in the world. As we will see, there's a full mapping between the behavior of idealized (perfectly rational) agents and certain mathematical structures of analysis. (This corresponds to the immersion step.) Using the resources of analysis, the relevant mathematical equations can then be solved (the derivation step). The results are then mapped back into the behavior of the idealized agents (through the interpretation step). However, there's no *full* mapping between the behavior of agents in the *actual* world and the same mathematical structures. After all, actual economic agents do not necessarily maximize their utility functions (assuming that there are such functions in the first place!). In this case, we have at best *partial* mappings between the behavior of actual economic agents (corresponding to the features that actual agents try to preserve in practice) and the relevant mathematical structures.

There are also partial mappings between the behavior of actual agents and *different* sorts of mathematical structures. This corresponds to a proposal advanced by Herbert Simon with the notion of "satisficing", according to which instead of trying to maximize profit, agents only try to get "satisficing" outcomes (see Simon 1982 and 1997). Given cognitive and computation limitations of actual agents, we will, once again, be dealing only with partial mappings, even though Simon's proposal is much less idealized than those articulated in neo-classical economics. Let's discuss these points in turn.

As is well known, neo-classical economics provides a simple, and highly idealized, model of rationality of economic agents in terms of maximization (for a critical examination, see Simon 1972, 409–410). In the theory of the firm, for instance, the goal is to *maximize profits*. Profit is characterized in terms of the difference between gross receipts from sales and cost of production. Two conditions should be met: (i) First, the *demand function* is introduced. According to that function, the quantity demanded (q_d) is a function of price; that is, $q_d = D(p)$. Since gross receipts (R) amounts to price times quantity, the demand function determines gross receipt; that is: $R = pq_d$. (ii) Second, the *cost function* is also introduced. According to that function, the cost of production (C) is a function of the quantity produced (q_s); that is, $C = C(q_s)$. Note that in setting up these conditions, we are *immersing* certain features of the empirical set up (in this case, certain aspects of the economic behavior of firms) into a given mathematical setting (in this case, part of mathematical analysis).

But to be able to *derive* any results from this setting, an additional idealized move has to be made. It's supposed that the quantity produced equals the quantity demanded, that is: $q_s = q_d$. As a result, the profit is just the difference between gross receipt and the cost of production. In symbols: Profit $= R - C = pq - C(q)$. The mathematical model now directly yields the results (*derivation* step). In particular, it's straightforward to determine when profit is maximized. All we have to do is to use a bit of analysis: $\mathrm{d}(R - C)/\mathrm{d}q = 0$.

The last step in the application process (the *interpretation* step) can be obtained by interpreting the results just obtained back into the original economic situation. A recommendation is then made with regard to the best combination of gross receipt and cost of production so that profit can be maximized.

Note that the economic agent is supposed to be able to solve the equations above. To do this, the agent must have perfect knowledge of all the data, and must be able to perform the necessary calculations without mistakes. There is, of course, a well-developed theory of maximization in analysis, and so there are plenty of resources to solve the relevant equations generated in the immersion step. As a result, representing the rationality of economic agents in terms of maximization is a convenient strategy to ensure that the derivation step can be met; that is, that the relevant equations can be solved. In fact, it's tempting to read this particular case of applied mathematics as one in which the tractability of the mathematics in the derivation stage has guided the application process—directing, in particular, the immersion step. After all, if economic agents are taken to maximize their utility functions, the corresponding equations, generated in the immersion step, can be solved through maximization techniques. It's then sensible to make the required idealization of regarding rationality in terms of maximization.[23]

However, as critics of neo-classical economics have pointed out, economic agents hardly (if ever) have perfect knowledge of all the data, mistakes are commonly made in mathematical calculations, and given the limitations in computational power and the complexity of environmental constraints, the agent may not be even able to perform the required derivations. Moreover, it's not clear whether actual economic agents even try to maximize their utility functions in the way suggested by the neo-classical conception. In other words, the maximization approach to economics seems to be *empirically inadequate*.

As a result, there are *no full mappings* between the *actual* economic situation and the mathematical models generated by neo-classical economics. After all, the latter models don't exactly describe the context in which economic agents actually find themselves. This doesn't mean, though, that neo-classical economics has nothing to offer. As even critics of neo-classical economics should be able to recognize, there are *partial* mappings between certain aspects of the actual economic scenario (e.g. the interest that actual firms have in increasing their profit) and the mathematical model (e.g. maximizing the profit function). The features of the actual economic scenario that have counterparts in the mathematical model (corresponding to R_1- and R_2-components) are mapped to that model via partial isomorphisms (or partial homomorphisms). This indicates in which respect neo-classical economics, although idealized, can still say something about the world, albeit indirectly. There are aspects of the actual world—although certainly not every aspect—that are successfully captured by the relevant models. Even in idealized contexts, there are partial mappings between empirical and mathematical structures.[24] Of course, this is not a defense of neo-classical economics—nor is it meant to be one. Our point is only to illustrate how the inferential conception can make sense of typical idealizations that are often involved in the process of applying mathematics.

But there is a different sort of response to neo-classical economics that is worth considering in the context of articulating an account of applied mathematics. As is well known, Herbert Simon has developed an alternative conception of rationality in economics, introducing the notion of bounded rationality (see Simon 1982, 203–494, and Simon 1997, 269–443). On his view, economic agents typically do *not* maximize their utility functions. They search for a *satisficing* alternative; that is, an alternative that is satisfactory given the constraints on the problem at hand. Simon's proposal emphasizes two basic sort of limitations involved in actual decision making in economics: (i) *limitations on the economic agent*: he or she has *limited* computational and cognitive power; and (ii) *limitations on the nature of information about the environment*: often, the agent has at best *incomplete information* about alternatives. Here are some examples. With regard to (i), given limitations on cognitive and computational power, the agent may not be able to solve certain equations necessary to determine the best course of action. With regard to (ii), there may be limited information about the behavior of the demand and the cost of production. (To accommodate mathematically this possibility, risk and uncertainty can be introduced in the demand function and in the cost function.) There are also cases in which both forms of limitation, (i) and (ii), are in place. For example, the complexity in the cost function and in other environmental constraints can be so large that it may prevent the agent from calculating the best course of action. Given all these forms of limitations, it becomes clear in which sense rationality is bounded.

In practice, by being sensitive to all of these features, Simon's conception leads to a dramatic re-conceptualization of the immersion step, which is now being guided by the *empirical set up* rather than by the ability to perform mathematical derivations. As a result, a different approach to the role of mathematics in economic theory emerges. (We'll return to this point below.) This re-conceptualization is only possible if we pay attention to the actual cognitive processes that economic agents go through when they make decisions. This is a key aspect of theories of bounded rationality. As Simons emphasizes:

> Theories of bounded rationality are more ambitious [than neo-classical theories] in trying to capture the actual process of decision as well as the substance of the final decision. A veridical theory of this kind can only be erected on the basis of *empirical knowledge of the capabilities and limitations of the human mind*; that is to say, on the basis of psychological research. (Simon 1987, 291; italics added.)

It's important to highlight two significant features of theories of bounded rationality: (i) The resulting economic theory is empirically adequate—or, at least, in a certain respect, it is more empirically adequate than neo-classical economics.[25] After all, the theory of bounded rationality incorporates assumptions about the *actual* behavior of economic agents and the limitations they face that neo-classical economics is silent about. (ii) Satisficing provides a sensible constraint on human rationality. Not only is it closer to the actual practice of economic agents, it's also sensitive to the actual difficulties that these agents face when they make economic decisions.

However, there are still difficulties with Simon's model. As we saw, neo-classical economists made a particular idealization about the behavior of economic agents (namely, that these agents attempt to maximize their utility functions) so that certain mathematical theories from analysis could be used to obtain the relevant derivations. With Simon's theory, that particular idealization is dropped. Nevertheless, it's unclear which mathematical theory could be used to obtain derivations regarding *satisficing* decisions (rather than maximizing ones). As a result, adjustments have now to be made at the mathematical level to obtain derivations that in the neo-classical proposal could be readily obtained through maximization techniques. Rather schematically, we could say that whereas in neo-classical economics we have a mathematical model in search of the actual economic world, in satisficing economics we have an economic world in search of a mathematical model.

In the end, however the mathematical adjustments required by Simon's proposal play out (see Simon 1987 and 1992 for details), there will be at best *partial mappings* between the actual economic situation and the new mathematical models, given that the economic situation is now thought of as fundamentally *incomplete*. As we saw, the situation is incomplete with regard to both the information that economic agents have *and* the agents' cognitive and computational capacities. Both sources of partiality (or incompleteness) can be accommodated by the inferential conception, using the partial structures framework. First, the limited information that economic agents have is accommodated in terms of partial relations: the R_1- and R_2-components of a partial relation correspond to the information that the agents actually have, whereas the R_3-component involves those chunks of information the agents fail to have so far. Second, the agents' computational *and* cognitive limitations can be represented by the partial mappings themselves (partial isomorphism and partial homomorphism). As we saw, these mappings only preserve *some* structure (and not all the structure) of the relevant context of investigation. This can be seen as a limitation in the *computational* capacity of the agents, who are only able to transfer a limited amount of information from the empirical set up to the mathematical model. Moreover, as we saw above, these partial mappings are *partial* functions, and so they aren't defined for every element in their domains. This can be seen as an expression of the limited *cognitive* capacity of the agents, who are able to know only a certain portion of the overall context of investigation. As a result, it's natural to expect that there will only be partial mappings from the relevant features of the actual economic situation to the relevant mathematical models.

To sum up, it now becomes clear that, also in Simon's case, partial isomorphisms or partial homomorphisms are central in the *immersion* step. These mappings allow us to move from the limitations of the empirical set up (the partiality of information that agents have and their limited cognitive capacities) to the appropriate mathematical models. Similarly, after *derivations* are obtained, the *interpretation* step can also be implemented with partial isomorphisms or partial homomorphisms. After all, as we saw, the empirical set up is now characterized by partial information, and so only partial mappings will hold from the mathematical model back to the empirical set up. In the end, partial mappings are crucial even when we deal with less idealized models, such as those developed by Simon.

5.4. Mathematics: unification, novel predictions, and explanation

How can the inferential conception accommodate the multiple roles that mathematical theories play in applied contexts? In particular, how can the conception make sense of the use of mathematical theories in the unification of otherwise disparate scientific theories, in making novel predictions, and in providing mathematical explanations? To answer these questions, the pragmatic and contextual factors involved in the application of mathematics must be taken into account. The inferential conception will highlight the function played by inferences in each of these roles. But to cash out the details in each case will require slightly different stories regarding the pragmatic and context-dependent factors, and ultimately, the story will depend on the commitment on the debate regarding realism and anti-realism in the philosophy of mathematics. However, as we will see, the inferential conception still manages to be neutral on this issue.

One way of thinking about the use of mathematics in unification, novel predictions, and mathematical explanations is in terms of epistemic accessibility, highlighting the function that inferences play in all of these issues. In fact, unification, novel predictions, and explanations are ultimately due to the *inferential role* of mathematics. As noted in Section 4 above, each of these three uses of mathematics can be formulated in terms of appropriate inferences between the relevant empirical and mathematical structures. Let's consider them in turn.

To *unify* different domains, inferential relations have to be established between them. For instance, by introducing complex numbers into the theory of differential equations, various solutions to these equations can be unified (for a discussion, see Colyvan 2002). This means that a multitude of apparently unrelated solutions to differential equations can be systematically brought together as part of a larger structure. How is this result achieved? In this case, we have two mathematical domains: (a) one is characterized by the theory of differential equations and a multitude of solutions to these equations; (b) the other involves the theory of complex numbers. The former domain looks rather disconnected, with the various solutions bearing almost no relation to each other. However, by bringing complex numbers into this domain, we can make sense of the way in which the various solutions are in fact connected to each other. The crucial work here is to bring, in the immersion step, some structure of the complex numbers into the theory of differential equations. As a result, in the derivation step, we can determine how various solutions to differential equations are related. In other words, this is a case where new inferential relations among solutions to differential equations are determined by complex numbers. Unification emerges here as the result of establishing such inferential relations among apparently unrelated things.

To provide *novel predictions*, inferences must also be established—particularly in the interpretation step. What is crucial here is to interpret suitably the mathematical structures so that relevant predictions can be made about the empirical set up. For example, when Dirac found negative energy solutions to the equation that now bears his name, he carefully devised *physically meaningful interpretations* of these solutions. He first ignored these negative energy solutions as non-physical, and took them to be just an artifact of the mathematics (as is commonly done in similar cases in classical mechanics). Later, however, he devised a physically meaningful interpretation to these negative energy solutions in terms of "holes" in a sea of electrons. But the resulting interpretation was empirically inadequate, since it entailed that protons and electrons had the same mass. Dirac finally devised yet another interpretation of the same mathematical formalism. He interpreted the negative energy solutions in terms of a new particle that had the same mass as the electron but opposite charge. A couple of years after publishing his results, the positron was detected (for further details and references, see Bueno 2005).

Note the role of non-structural pragmatic and contextual considerations here. First, taking the negative energy solutions as non-physical is a pragmatically motivated interpretation of the formalism. And it goes beyond the formalism in a non-structural way, since it denies that certain features of the mathematical structure represent aspects of the physical world. Second, taking the negative energy solutions to stand for "holes" in a sea of electrons emerged in the context of accepted theoretical results—in particular, Pauli's exclusion principle—and it initially made perfect sense in light of such results. Finally, the rejection of the "hole" interpretation for the positron interpretation was motivated by extra-structural, contextual considerations. In particular, the need for the interpretation to be empirically adequate in the context of accepted theoretical results about the mass of protons and electrons motivated Dirac to develop the positron interpretation. And it's worth explicitly noting that all of the interpretations in question are mappings from a mathematical structure to the physical set up, and they are not uniquely determined by the structure alone—hence the need for pragmatic and contextual considerations in the selection of suitable mappings.

This example illustrates that to obtain novel predictions from a mathematical framework, the crucial work is done in the interpretation step. Dirac used the same equation to generate two radically different physical interpretations: one turned out to be empirically inadequate, but the other played a central role in the (novel) prediction of the positron. Here, once again, the key move is to establish, as part of the interpretation step, appropriate inferential relations.

This example also illustrates how the inferential conception explains which parts of the mathematical models refer and which do not. First, the conception provides a framework to locate and conceptualize the issue: the work is ultimately done at the interpretation step. Some interpretations are empirically inadequate, and thus fail to provide an entirely successful account of the application process. Dirac's interpretation of the negative energy solutions as 'holes' clearly illustrates this point. However, despite being at best only partially successful, such empirically inadequate interpretations can be very helpful in paving the way for empirically successful interpretations. They offer some understanding of how the world could be if the interpretation were true, and they can lead the way to interpretations that are empirically supported. Again, Dirac's interpretation of the negative energy solutions in terms of the 'positron' beautifully illustrates this point. As a result, the inferential conception sheds light on the issue of how mathematical models with non-referring elements can be useful. Although the inferential conception cannot provide a recipe for identifying which interpretations will eventually be empirically successful—and it's unclear that we should expect that any proposal could deliver that—the account offers a framework in terms of which the various options can be suitably represented and assessed.

Finally, to articulate *mathematical explanations* it's also crucial to establish inferential relations between mathematical structures and the (suitably interpreted) empirical set up. The key inferential moves emerge here in the immersion and in the interpretation steps. For example, we mentioned above the case of the standard Lotka-Volterra explanation of why all populations whose abundance exhibits cycles must be part of a predator–prey pair. The mathematical explanation emerges from three mathematical facts: (a) there are no periodic solutions to first-order differential equations; (b) coupled first-order differential equations are equivalent to a second-order differential equation, and (c) second-order differential equations allow periodic solutions. These three facts are part of the mathematical structure invoked to generate the need for a predator–prey pair in the case of populations whose abundance exhibits cycles. Of course, the key move here is to establish appropriate inferential relations (in this case, by finding suitable mappings) between the biological domain and the mathematical structure, so that the relevant explanation can be obtained. In particular, one needs to establish a biologically significant interpretation of periodic solutions, and a mathematically sound reading of the predator–prey pair. The latter is achieved in the immersion stage, and the former in the interpretation step. In other words, attention is needed both in the immersion and in the interpretation stages, so that the relevant mathematical facts (namely, (a)–(c), above) can be invoked to yield the appropriate derivations.

These considerations sketch how the inferential conception deals with three central issues that the mapping account failed to do justice to. As we saw, what mathematical unification, novel predictions by mathematical reasoning and mathematical explanations have in common are the various *inferential roles* played by mathematics. In some instances, mathematics provides additional entities to quantify over. This was the case of the use of complex numbers to establish new inferential relations among solutions to differential equations. In other cases, mathematics may have a heuristic value, in the sense that the mathematical formalism is the source of interpretations that are physically meaningful—and these interpretations generate, in turn, significant inferences, namely, novel predictions. This was the case of Dirac's inference regarding the positron. And in yet other cases, mathematics has an explanatory role, and again this is accomplished by establishing inferential relations between suitably interpreted mathematical structures and empirical phenomena. This was the case of the Lotka-Volterra explanation.

Can both realists and anti-realists about mathematics adopt the inferential conception? We think so. What will be different between realist and anti-realist accounts of mathematics is the way in which each view interprets the success in obtaining the inferences discussed above. For the platonist, inferences used in the successful unification of different (mathematical) theories, or in the prediction of novel phenomena via the (indispensable) use of mathematical theories, or in the mathematical explanation of phenomena support the realist commitment to the corresponding mathematical entities. After all, quantifying over the relevant entities is indispensable to obtain the results in question. In every step of the application process, from the immersion through the derivation to the interpretation stages, mathematical entities are invoked, and thus, the platonist insists, we are ontologically committed to these entities.

For the nominalist, however, inferences used in all of these three contexts (of unification, novel prediction, and explanation), despite the quantification over mathematical objects, need not be taken as committing one to the *existence* of the corresponding objects. There are two reasons for this. First, the nominalist may distinguish quantifier commitment and ontological commitment, and may insist that, *pace* Quine, quantification is not enough for ontological commitment (see Azzouni 2004). The fact that quantifiers can be used without ontological import is commonly recognized. If someone asserts that "There was a detective who lived in 221B Baker Street in London, and who always surprised his friend Watson by solving crimes brilliantly", they are not therefore committed to the existence of this detective (Sherlock Holmes). We acknowledge very naturally, for instance, in the case of fictions, the distinction between quantification over an object and commitment to the existence of that object. And with this distinction in place, the nominalist can state that despite quantifying over mathematical objects, he or she is not committed to the *existence* of mathematical entities. An existence predicate would need to be met in order for us to make ontologically committing claims (see Azzouni 2004 and Bueno 2005, and also Colyvan 2005 for criticism of such an approach).

Second, the nominalist may argue that, in applied mathematics, what is crucial is to make sense of the mathematical formalism in a physically significant way. For the nominalist, according to the inferential conception, both the immersion and the interpretation steps in the application process presuppose a particular physical interpretation of the mathematical formalism. This is illustrated, for instance, in the case of Dirac's work, by the different physical interpretations he assigned to the same mathematical formalism. Moreover, the nominalist can accommodate the derivation step without commitment to the existence of mathematical entities, by cashing out the notion of consequence in modal terms. That is, the nominalist would insist that A is a logical consequence of B if, and only if, necessarily, if B then A—where the necessity operator is defined in the usual way from a primitive notion of logical consistency, and the conditional is a material one (for details, see Field 1989). For these two reasons, ontological commitment, for the nominalist, is ultimately restricted to the physics rather than the mathematics (see Bueno 2005, and Bueno forthcoming).

Suppose, however, that the nominalist adopts the more standard line of accepting Quine's identification of quantifier commitment and ontological commitment in the case of entities that are indispensable to our best scientific theories. In this case, the nominalist will have to rewrite the relevant scientific theories to establish that they don't involve commitment to mathematical objects (in the way articulated, for instance, by Field 1980). In this instance, it should be noted that unifications, explanations, predictions, and various inferences may be more transparent (epistemically) in the platonist version of the theory than in the nominalist version of the theory. However, these same unifications, explanations, and so on must also be available in the nominalist version—if mathematics is conservative. The idea is that the so-called pragmatic virtues of unification, explanation and the like are, according to the view under consideration, taken to be merely results of certain inferences. If the mathematics used to tease out these inferences is conservative, as Field (1980) argues, then the results of these inferences could be derived in the nominalistic version of the theory. The derivations in this case will typically be less transparent without the mathematics, but there should be no results that could not be obtained by nominalistic means (granting conservativeness of the mathematical theories in question).[26]

5.5. Some complications and illustrations

Let us now briefly consider some complications for the account suggested here. Addressing these complications will also give us an opportunity to discuss some additional examples that illustrate how the inferential conception works.

First, we consider the case of composite functions. As we mentioned previously, sometimes the applications of mathematics will be a multi-stage process where the empirical system is modeled in one mathematical structure, which in turn may be modeled in another mathematical structure. Such complications present little difficulty for the inferential conception of mathematics. We just need to ensure that the composite mappings in question are defined (e.g., the range of the first is in the domain of the second). An example will help illustrate the idea. In population ecology, population abundance is discrete (indeed, it's natural-number valued). But for many reasons (including the convenience of being able to utilize the machinery of differential equations), it turns out to be useful to represent population abundance with a real-valued function. This can be thought of as a two-stage mapping at the interpretation stage. At the immersion stage, we map from population abundance to the real numbers, but where the range of this mapping is confined to a subset of the natural numbers. After derivations, we may find solutions that are not natural numbers. Here we invoke a two-stage interpretation mapping. First, we map from the continuous to the discrete by familiar modularizations, then we map from the discrete mathematics to the population. We note that although non-natural number values for population abundances have no direct physical significance, they are not uninterpretable. We need to do little more than round off to the nearest natural number and interpret the result as a population abundance.

But not all examples are so simple. Sometimes, we end up with results that seem to defy obvious physical interpretation and yet we may at least wish to entertain the idea of them being physically significant. The problem in such cases is that the immersion mapping is not invertible and there is no obvious two-stage process (as in the last example) to enable the obvious interpretation. Indeed, part of the problem here is that there is no obvious interpretation.[27] What we require is an invertible mapping that is a *conservative extension* of the mapping used at the immersion stage. That is, we require an invertible mapping that agrees with the immersion mapping on all cases, so provides an interpretation of all the non-problematic cases (i.e., those which have physical interpretations under the original interpretation mapping), but which also provides physical interpretations of the problematic cases. Typically, such conservative extensions will not be unique, but some will be more plausible than others. Context and pragmatic considerations will come into play in reducing the number of choices. In some cases, more than one interpretation might require serious consideration. It is not clear that there is a general account of how the construction of these conservative extensions of existing interpretations will proceed. The details, very likely, will depend on the context, and will require a case-by-case treatment.

But despite such difficulties, the inferential conception of applied mathematics does have the resources to deal with such problem cases. It does this by (i) allowing pragmatic and contextual factors to play a role, and (ii) by being less restrictive about the kinds of mappings that might be invoked at the immersion and interpretation stages. In this case, we're allowing the construction of new interpretation mappings, so long as they are "conservative extensions" (in the sense specified above) of the existing interpretation mapping.

6. Conclusion

For the reasons discussed above, the mapping account of the applicability of mathematics is an interesting first step. But before it can be considered a full and correct account of mathematics in empirical applications, much more needs to be said about the kind of mappings and about cases where mathematics seems to be doing more than merely representing. This is particularly so in the cases where mathematics might be thought to be providing explanations of certain aspects of the empirical systems being modeled.

As an alternative conception that incorporates the correct structural features found in the mapping account, we offer an inferential conception of the application of mathematics. According to this conception, applied mathematics does involve crucial mappings between mathematical structures and the empirical set up. But crucial pragmatic and context-dependent factors also play a central role.

A final question concerns the relationship between the mappings (in either the mapping account or those in the inferential account) and the success of a mathematical application. For instance, are such mappings either necessary or sufficient for a mathematical structure to be applied fruitfully to the world?[28] Let's take the sufficiency question first. The answer here is clearly "no". It is not sufficient for a successful application that there are mappings. As we've alluded to in passing, some mappings will not be useful and may even mislead. There are bad mathematical models and these too can be represented in our framework. What's the difference between the good models and the bad ones—apart from the trivial difference that the good models get things right, or are at least useful? There is no non-trivial answer to this question. Good modeling is something of an art, and leans heavily on contextual and pragmatic considerations.

The more interesting question is whether mappings are necessary for a successful application of mathematics. If what we have argued in this paper is correct, the inferential account provides *one* way to understand applications of mathematics, and in particular, the successful applications. For all we have said thus far, it is an open question whether invoking mappings is the *only* way to make sense of mathematical applications. Although we suspect that it is necessary to invoke mappings, we do not pretend to have established that here. We are content to rest with our more modest conclusion that mappings, as employed in the inferential conception, can provide a satisfying account of applied mathematics.

Notes

[†] To contact the authors, please write to: Otávio Bueno, Department of Philosophy, University of Miami, Coral Gables, FL 33124-4670, U.S.A., e-mail: otaviobueno@mac.com; Mark Colyvan, Department of Philosophy, University of Sydney, Sydney, NSW 2006, Australia, e-mail: mcolyvan@usyd.edu.au.

[1] As will become clear below, to read off facts about the physical world from a mathematical theory, it's crucial to provide suitable physical interpretations of the mathematical formalism. Without such interpretations, it's unclear how mathematical theories alone could have any implications for the physical world—except, perhaps, regarding the cardinality of the domain of the empirical set up. For example, if a certain mathematical theory only has finite models, and the empirical set up has infinitely many objects, the empirical set up will be considered mathematically impossible by the mathematical theory. Alternatively, the mathematical theory will be considered inadequate to represent correctly the empirical domain.

[2] At this point, the analogy between maps and mathematical theories may not go through completely, since in the case of mathematical theories, there's often more structure at the mathematical level than in the empirical setting. We will return to this point below.

[3] Here we assume that the world comes with a particular instantiation of the relevant objects and their structural relations (e.g. billiard balls and causal connections).

[4] Pincock (2004b, pp. 150–155) does discuss the issue of whether the mappings in question are themselves mathematical objects, but that is not the issue here. We are concerned with the

kind of map, that is, the relationship between the target system and the mathematical model of the system.

[5] We are grateful to an anonymous referee for raising this issue.

[6] Recall that a *homomorphism* is a mapping from one structure A (with a domain D, and a family of relations R among the elements of D) to another structure B (with codomain D', and a family of relations R' among the elements of D') that respects the relations of A by assigning each element of R to a corresponding element of R'. That is, a homomorphism maps not only the objects of one domain to another; it does so in such a way that preserves certain aspects (although typically not all) of the structures involved. An *epimorphism* is a surjective homomorphism (i.e., every member of the codomain is the image of at least one member of the domain); a *monomorphism* is a injective homomorphism (i.e., different members of the domain are mapped to different members of the codomain).

[7] Here and in what follows we take W to be the *assumed structure* and that we have somehow dealt with the issue of how to determine what the assumed structure is.

[8] To give a very simple example, consider a map that takes objects to their masses. Now consider two objects a and b whose individual masses are each 1kg. The map in question maps a to 1 and b to 1. Of course, we can derive in the mathematics $1 + 1 = 2$, and intuitively we want to say that this shows that the combined mass of the two objects is 2kg. But we cannot say this without an interpretation (or inverse map) of the mathematics. We are not insisting on a unique single inverse map, just that there be some suitable set of partial inverse maps. Without this we cannot interpret the derived mathematical result as saying anything about the target system. We use this simplistic example as an illustration only. There is no need for anything as sophisticated as the mapping account (or the inferential account we will develop later in the paper) to explain what's going on in a straightforward application of arithmetic. After all, all of this can be done in first-order logic. The point is simply that without an inverse mapping the mathematics remains uninterpreted and says nothing about the empirical system it is supposed to be representing.

[9] In some cases, the non-physical solution will be the launch place of the projectile. But in other cases, the second solution will have no physical interpretation at all (if the launch place is on a cliff above a plain, for instance).

[10] Pincock explicitly acknowledges this incompleteness (Pincock 2004b, p. 137), but feels that a necessary first step is to provide an account of non-idealized mathematics. In this regard, we can be seen to be extending Pincock's account.

[11] It may well be that a mathematical analysis of some representational system can offer explanations, but even this is a significant departure from the mapping account. According to the mapping account, mathematics is merely a representational tool, and any explanations that drop out of the mathematics must be just standing proxy for the real physical explanation.

[12] Consider an explanation for why it takes so long on foot to cover the short distance between Campo San Angelo and Campo San Polo in the Italian city of Venice. Even a cursory glance at the street map shows the chaotic street layout carved up by the various canals. Add to this the fact that the map shows Campo San Angelo on one side of the Grand Canal and Campo San Polo on the other side with very few bridges over the Grand Canal (and none directly between Campo San Angelo and Campo San Polo). But this is not really an explanation based on the map. It is just a case of the map making plain certain facts about the city itself. The *real* explanation of the time it takes to get from Campo San Angelo to Campo San Polo is in terms of the city's canals, chaotic streets, and lack of bridges over the Grand Canal. The street map does not explain—facts about the city do.

[13] See Colyvan (2001a and 2007), Baker (2005), and Lyon and Colyvan (2008) for further examples, and Ginzburg and Colyvan (2004) for more on the mathematical explanation of population cycles.

[14] It is perhaps worth mentioning that much of the motivation for Pincock (and for Leng as well) is to defuse the indispensability argument for mathematical realism. Although we think that

the problems with the mapping account and our suggested positive proposal have ramifications for this debate in the philosophy of mathematics, we won't pursue these matters here. We intend to take up the issue elsewhere.

[15] Inferences are typically understood in terms of truth-preservation among propositions. It may appear that our usage of the term is different from this in so far as we are talking about inferences from maps, mathematical structures and the like. But this is not right. The idea here is that the mapping carries the inferential structure from one domain to the other. To give a particular well-known example, two isomorphic structures are elementarily equivalent (and so the same first-order sentences that are true in one structure are also true in the other). This structural similarity generates an inference procedure that allows us to infer from the fact that a certain result holds in one structure that it also holds in the other. In light of this, there is nothing deviant about our apparently more general usage of the notion of inference.

[16] Inference is also crucial in scientific representation more generally (see Suárez 2004).

[17] Just as in the mapping account, we take the empirical set up to have an *assumed structure*. That is, we assume that there is some natural structure of the set up in question or that an appropriate structure can be imposed upon the set up. We do not assume that this is trivial nor that it will be unique, although often there will be some natural candidates for the structure in question. As we shall see in section 5.1, however, the inferential account is better equipped to deliver a suitable assumed structure.

[18] See Hughes 1997 for a related account of scientific representation in terms of denotation, demonstration, and interpretation. The inferential conception of applied mathematics provides an extension of Hughes' account of scientific representation to the application of mathematics. (For a different extension of Hughes' proposal in the context of nanoscale research, see Bueno 2006.)

[19] Even in the simple example just presented of calculating the combined mass, we might have proceeded via a more complicated route. We might, for example, have invoked momentum and calculated the mass by considering the relationship between mass and momentum. Moreover, in some contexts, this may well be the most efficient way to proceed, for example, if we cannot directly determine the mass.

[20] See Batterman (2002) for further details of this and other examples. The relationship between Batterman's asymptotic reasoning and the inferential account of applied mathematics is a topic that we plan to explore elsewhere.

[21] Indeed, the fact that in the inferential account the interpretation step is not the inverse of the immersion step is already an acknowledgement of fluidity in the assumed structure.

[22] Note that both partial isomorphism and partial homomorphism are *partial* functions, and so they are not defined for every element in their domains. This point will be important for our discussion below.

[23] It might be argued that the agent does not need to solve the equations in question, depending on the kind of model under consideration. For instance, a purely descriptive model (one intended to describe the actual behavior of a real agent and without offering explanations of their behavior) can be a good model even if the agent never (consciously) tries to maximize expected utility or the like. The model works in the sense that it gets the behavior right but that's all. If, on the other hand, the model is supposed to not only model the agent's behavior but get their reasons for such behavior right, then the model better reflect the actual cognitive processes. In this latter case, the agent would need to be a conscious expected utility maximizer.

[24] For an examination of partial mappings and idealization in physics, see Bueno, French, and Ladyman 2002, and French and Ladyman 1998.

[25] See Bueno 1997 for a formal account of degrees of empirical adequacy.

[26] There are some substantial issues here about the nature of explanation. Must an explanation be epistemically accessible? Does mere knowledge that the appropriate connections can be established count as an explanation? Are some explanations non-causal? Can mathematics

provide explanations of physical phenomena that do not merely detail causal histories? (See Colyvan 2001a, and 2007 for further discussion of issues.)

[27] We have in mind here Dirac's puzzle of the interpretation of negative energy solutions to Dirac's equation. As we noted above, the interpretation of these solutions led to the discovery of the positron.

[28] Our thanks go to an anonymous referee for raising this question.

References

Ahlfors, Lars V. (1979), *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*. New York: McGraw Hill.

Azzouni, Jody (2004), *Deflating Existential Consequence: A Case for Nominalism*. New York: Oxford University Press.

Baker, Alan (2003), "The Indispensability Argument and Multiple Foundations for Mathematics", *Philosophical Quarterly* 53: 49–67.

——— (2005), "Are there Genuine Mathematical Explanations of Physical Phenomena?", *Mind* 114: 223–238.

Balaguer, Mark (1998), *Platonism and Anti-Platonism in Mathematics*. New York: Oxford University Press.

Batterman, Robert W. (2002), *The Devil in the Details: Asymptotic Reasoning in Explanation Reduction and Emergence*. New York: Oxford University Press.

Bueno, Otávio (1997), "Empirical Adequacy: A Partial Structures Approach", *Studies in History and Philosophy of Science* 28: 585–610.

——— (2005), "Dirac and the Dispensability of Mathematics", *Studies in History and Philosophy of Modern Physics* 36: 465–490.

——— (2006), "Representation at the Nanoscale", *Philosophy of Science* 73: 617–628.

——— (forthcoming), *The Dispensability of Mathematics*, unpublished manuscript, forthcoming.

Bueno, Otávio, Steven French, and James Ladyman (2002), "On Representing the Relationship between the Mathematical and the Empirical", *Philosophy of Science* 69: 497–518.

Cartwright, Nancy (1983), *How the Laws of Physics Lie*. Oxford: Clarendon Press.

Colyvan, Mark (2001a), *The Indispensability of Mathematics*. New York: Oxford University Press.

——— (2001b), "The Miracle of Applied Mathematics", *Synthese* 127: 265–278.

——— (2002), "Mathematics and Aesthetic Considerations in Science", *Mind* 111: 69–74.

——— (2005), "Ontological Independence as the Mark of the Real", *Philosophia Mathematica (3)* 13: 216–225.

——— (2007), "Mathematical Recreation Versus Mathematical Knowledge", in M. Leng, A. Paseau, and M. Potter (eds.), *Mathematical Knowledge*. Oxford: Oxford University Press, 109–122.

da Costa, Newton, and Steven French (2003), *Science and Partial Truth*. New York: Oxford University Press.

Field, Hartry (1980), *Science without Numbers*. Princeton, NJ: Princeton University Press.

——— (1989), *Realism, Mathematics and Modality*. Oxford: Blackwell.

French, Steven, and James Ladyman (1998), "A Semantic Perspective on Idealization in Quantum Mechanics", in N. Shanks (ed.), *Idealization in Contemporary Physics*. Amsterdam: Rodopi, 51–73.

Ginzburg, Lev, and Mark Colyvan (2004), *Ecological Orbits: How Planets Move and Populations Grow*. New York: Oxford University Press.

Hughes, R.I.G. (1997), "Models and Representation", *Philosophy of Science* 64: S325–S336.

Leng, Mary (2002), "What's Wrong with Indispensability? (Or the Case for Recreational Mathematics)", *Synthese* 131: 395–417.

Lorentz, H.A. (1904), "Electromagnetic Phenomena in a System Moving with any Velocity Less than that of Light", reprinted in A. Einstein, H.A. Lorentz, H. Weyl and H. Minkowski (eds.), *The Principle of Relativity*. New York: Dover, 1952.

Lyon, Aidan, and Mark Colyvan (2008), "The Explanatory Power of Phase Spaces", *Philosophia Mathematica* (3) 16: 227–243.

Pincock, Christopher (2004a), "A Revealing Flaw in Colyvan's Indispensability Argument", *Philosophy of Science* 71: 61–79.

——— (2004b), "A New Perspective on the Problem of Applying Mathematics", *Philosophia Mathematica* (3)12: 135–161.

——— (2007), "A Role for Mathematics in the Physical Sciences", *Noûs* 41: 253–275.

Resnik, Michael D. (1997), *Mathematics As a Science of Patterns*. Oxford: Clarendon Press.

Shapiro, Stewart (1997), *Philosophy of Mathematics: Structure and Ontology*. New York: Oxford University Press.

Simon, Herbert (1972), "Theories of Bounded Rationality", reprinted in H. Simon, *Models of Bounded Rationality*, vol. 2, Cambridge, Massachusetts: MIT Press, 1982, 408–423.

——— (1982), *Models of Bounded Rationality*, 2 vols. Cambridge, Massachusetts: MIT Press.

——— (1987), "Bounded Rationality", reprinted in H. Simon, *Models of Bounded Rationality*, vol. 3. Cambridge, Massachusetts: MIT Press, 1997, 291–294.

——— (1997), *Models of Bounded Rationality*, vol. 3. Cambridge, Massachusetts: MIT Press.

Steiner, Mark (1998), *The Applicability of Mathematics as a Philosophical Problem*. Cambridge, M.A.: Harvard University Press.

Suárez, Mauricio (2004), "An Inferential Conception of Scientific Representation", *Philosophy of Science* 71: 767–779.

Wigner, Eugene P. (1960), "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", *Communications on Pure and Applied Mathematics* 13: 1–14.