Fictionalism in the Philosophy of Mathematics

Fictionalism in the philosophy of mathematics is the view that mathematical statements, such as ‘8+5=13’ and ‘π is irrational’, are to be interpreted at face value and, thus interpreted, are false. Fictionalists are typically driven to reject the truth of such mathematical statements because these statements imply the existence of mathematical entities, and according to fictionalists there are no such entities. Fictionalism is a nominalist (or anti-realist) account of mathematics in that it denies the existence of a realm of abstract mathematical entities. It should be contrasted with mathematical realism (or Platonism) where mathematical statements are taken to be true, and, moreover, are taken to be truths about mathematical entities. Fictionalism should also be contrasted with other nominalist philosophical accounts of mathematics that propose a reinterpretation of mathematical statements, according to which the statements in question are true but no longer about mathematical entities. Fictionalism is thus an error theory of mathematical discourse: at face value mathematical discourse commits us to mathematical entities and although we normally take many of the statements of this discourse to be true, in doing so we are in error (cf. error theories in ethics).

Although fictionalism holds that mathematical statements implying the existence of mathematical entities are strictly speaking false, there is a sense in which these statements are true—they are true in the story of mathematics. The idea here is borrowed from literary fiction, where statements like ‘Bilbo Baggins is a hobbit’ is strictly speaking false (because there are no hobbits), but true in Tolkien’s fiction *The Hobbit*. Fictionalism about mathematics shares the virtue of ontological parsimony with other nominalist accounts of mathematics. It also lends itself to a very straightforward epistemology: there is nothing to know beyond the human-authored story of mathematics. And coming to know the various fictional claims requires nothing more than knowledge of the story in question. The most serious problem fictionalism faces is accounting for the applicability of mathematics. Mathematics, unlike Tolkien’s stories, is apparently indispensable to our best scientific theories and this, according to some, suggests that we ought to be realists about mathematical entities.

It is fair to say that there are serious difficulties facing all extant philosophies of mathematics, and fictionalism is no exception. Despite its problems, fictionalism remains a popular option in virtue of a number of attractive features. In particular, it endorses a uniform semantics across mathematical and non-mathematical discourse and it provides a neat answer to questions about attaining mathematical knowledge. The major challenge for fictionalism is to provide an adequate account of mathematics in applications.

1 Motivation for fictionalism

There are two competing pressures in finding an adequate philosophy of mathematics (Benacerraf 1983). The first is to provide a uniform semantics across mathematical discourse and non-mathematical discourse. We want sentences such as '8 is larger than 5' to be treated semantically in the same way as sentences such as 'Sydney is larger than San Francisco', for at face value they seem to have the same structure and ought to have similar truth conditions. The second pressure is to provide an adequate naturalistic epistemology, one that does not make a mystery of how we come by mathematical knowledge. Why these are usually taken to be competing pressures is that realist philosophies of mathematics have little problem providing a uniform semantics but typically have trouble providing a naturalistically acceptable epistemology. Nominalist philosophies of mathematics, on the other hand, typically have difficulty providing a uniform semantics, with many nominalist philosophies having to give up on this entirely (see REALISM AND ANTIREALISM IN MATHEMATICS). But nominalist accounts fare much better with epistemology, for according to these theories mathematical knowledge—whatever it is—is not knowledge of abstract entities (see ABSTRACT OBJECTS).

Fictionalist philosophies of mathematics can be seen to be providing an elegant way of dealing with these two competing pressures. Fictionalism does employ a uniform semantics. '8 is larger than 5' is read at face value in the obvious way just as 'Sydney is larger than San Francisco' is. The difference, according to fictionalism, is that the latter sentence is true but the former is false. '8 is larger than 5' is taken to be false because there are no referents for '8' and '5'. But the semantics in both cases are the same. As with other nominalist theories of mathematics, epistemology does not present any serious difficulties for fictionalism. According to fictionalism, there is no mathematical knowledge apart from knowledge of the fiction of mathematics itself. Knowing that in the story of mathematics 2+3 = 5 is no more problematic than knowing that in the Tolkien story Bilbo Baggins is a hobbit. In both cases we know this by reading the relevant stories, listening to others who are well versed in the stories in question or, more adventurously, by exploring the logical consequences of the respective stories (see FICTIONAL ENTITIES).

The price of fictionalism, however, is the unintuitive claim that much of mathematics is false. (I say 'much of mathematics' because fictionalism does preserve the truth values of negative existentials like 'there is no largest prime number'. This statement is true in standard mathematics, and therefore true in the story of mathematics, but according to fictionalism it is also true simpliciter, because there are no numbers and *a fortiori* there is no largest prime.) While at first it might seem unintuitive to claim that '2+3 = 5' is false, but to claim otherwise is to commit oneself to the existence of numbers. After all, it follows straightforwardly from '2+3 = 5' that, for example, there exist numbers x, and y, such that $x+y = 5$. According to fictionalists, the existence of mathematical objects is problematic enough to warrant denying the truth of such statements.

It is also important to note that fictionalism in mathematics does not mean that 'anything goes'. Authors of mathematical theories, like writers of good literary fiction, are not free to develop their fiction in any way they please. For a start, consistency is usually thought to be strongly desirable. Beyond that, there are also requirements not to introduce unnecessary items. In good mathematics, like good literary fiction, posited entities contribute to the story. But perhaps the greatest constraint on writing mathematical fiction is that the latest instalment must be consistent with all previous instalments. Previous generations of mathematicians introduced such "characters" as sets, functions, natural numbers and so on. The current generation of mathematicians must develop these "characters" in ways that are consistent with what went before. It is as though current mathematicians are all contributing to a multi-authored series of books. Just as Tolkien was heavily constrained in the last book in *The Lord of the Rings* trilogy by what went before in *The Hobbit* and the previous two books in *The Lord of the Rings* series, so too modern mathematicians cannot develop the fiction of mathematics in any way they please.

There are several problems associated with admitting mathematical entities into one's ontology. First, accepting mathematical entities would seem to run into trouble with Ockham's razor. This is the advice not to multiply entities beyond necessity. It would appear that nominalists of all varieties have Ockham on their side, since they do not need to posit the huge number of entities entertained by Platonists. Second, mathematical entities are epistemically suspect. Mathematical entities are usually taken to be abstract, in the sense that they do not exist in space and time and do not have causal powers. It is thus mysterious how we can have knowledge of such causally isolated entities (Benacerraf 1983); or at least, an account is required of how the methods of mathematics are reliable means of forming beliefs about such abstract entities (Field 1989). Finally, we might add a more general worry about the metaphysical dubiety of abstract entities. After all, entities not located in space and time and without causal powers are utterly unlike any other entities we know about. These are the main motivations for fictionalism and are why fictionalists are prepared to give up on the truth of mathematics. It is the price they must pay in order to avoid a commitment to mathematical objects and the problems arising from accepting such entities.

2 The challenge for fictionalism

The biggest problem facing fictionalism in mathematics is to explain the central role mathematics plays in scientific inquiry. According to some (Quine 1981, Putnam 1971, Colyvan 2001) there is a powerful argument for the existence of mathematical entities that needs to be confronted. This argument draws upon the indispensable role mathematics plays in our best scientific theories. According to defenders of this line of thought, we ought to believe our best scientific theories, and that entails believing in the entities indispensably posited by those theories. And since mathematical entities would seem to be such entities, this means that we ought to be committed to the existence of mathematical entities. This argument is known as *the indispensability argument* and has been the focus of a great deal of recent work in the philosophy of mathematics. In particular, nominalists have scrutinised the argument, looking for ways to avoid the (for them) unpalatable conclusion. It is fair to say that all nominalists need a response to this argument.

Before discussing the various fictionalist responses to the indispensability argument, it will be useful to lay out the argument rather more carefully.

Premise 1: We ought to believe in all and only the entities of our best scientific theories.

Premise 2: Mathematical entities are indispensable to our best scientific theories.

Conclusion: We ought to believe in mathematical entities.

A couple of things are worth noting about the argument as presented here. First, it is supposed to mimic a style of argument endorsed by scientific realists. According to scientific realists, we are committed to electrons, black holes and other unobservable theoretical entities because of the role those entities play in our best scientific theories. The argument might be thought of as an attempt to push scientific realists a bit further—to mathematical realism—but might have little force with scientific anti-realists. Premise 1, in particular, appeals to a certain naturalistic attitude, whereby we are encouraged to look towards science for answers to questions about ontology. This much is typically endorsed by scientific realists. But Premise 1 says more: it also suggests that we ought to believe in *all* the entities of our best scientific theories. This gives voice to a kind of holism about scientific theories, whereby we cannot pick and choose among the parts of our best scientific theories. According to the holism in question, we believe our best confirmed theories in their entirety. The kind of holism involved seems to be confirmational holism, which holds that theories are confirmed or disconfirmed as wholes, not one hypothesis at a time.

With the indispensability argument in this form and these few clarificatory remarks in place it is clear that fictionalists have two basic options: deny Premise 1 or deny Premise 2. The first option typically involves giving up holism, although usually this is attempted while trying to maintain a commitment to naturalism and scientific realism. The second option involves showing that, in the relevant respect, mathematics is dispensable to science. The latter brings us to Hartry Field's (1980) heroic attempt to do science without numbers.

3 Field's fictionalism

Field's project is motivated by a commitment to providing a uniform semantics and by epistemological concerns with Platonism. (The particular approach Field adopts is also motivated by a couple of other considerations: a commitment to providing intrinsic explanations (i.e. explanations that do not rely on extraneous entities), and the elimination of arbitrariness from scientific theories (e.g. the elimination of conventional co-ordinate frames and units of distance). While these considerations provide the motivation for fictionalism, according to Field, an adequate fictionalist philosophy of mathematics must explicitly address the indispensability argument. For the latter he takes to be the only good argument for Platonism, and as such it presents a serious obstacle to any nominalist philosophy of mathematics: undermine the indispensability argument and you undermine Platonism. It is Field's willingness to take the indispensability argument head on that gives his account its distinctive flavour.

According to all varieties of mathematical fictionalism, most of accepted mathematics is strictly-speaking false, but is true in the fictional story of mathematics. But Field recognises that the fictionalist account cannot stop there. After all, why should this particular fiction—the fiction of standard mathematics—prove to be in such demand in science? Field's answer to this question is ingenious. He simultaneously suggests how mathematics might be dispensed with and how, despite its dispensability, it could be used so fruitfully in science.

The first part of Field's project—showing the dispensability of mathematics—begins by showing how a typical scientific theory such as Newtonian gravitational theory might be constructed without quantifying over mathematical items. The basic idea is to be a substantivalist about space-time (see SPACETIME) and then work directly with space-time points/regions. Instead of talking of the gravitational potential, for example, of some space-time point, Field compares space-time points with respect to their gravitational potential. The former standard way of talking (in terms of gravitational potential of space time points) involves a gravitational potential function which is a map from the space-time manifold to real numbers and this seems to commit one to realism about space, time, functions, and the real numbers. But Field, following a suggestion of the mathematician David Hilbert, notices that one can do all one wants merely by comparing space-time points with respect to their gravitational potential. This relational approach does away with the nominalistically unacceptable mathematical machinery (functions and real numbers) in the theory itself. But Field also proves a representation theorem (Field 1980, 55–91) that shows that in the meta-theory one can recover all the relevant numerical claims. In particular, in the space-time theory Field considers (a fragment of Newtonian gravitational theory), there are no gravitational potential functions, mass-density functions or spatio-temporal coordinate functions, but the representation theorem guarantees that these are recoverable in the meta-theory. So, in a sense, nothing is lost.

It is important to note that Field does not advocate doing science without mathematics; it is just that science *can* be done without mathematics. And the latter is enough to suggest that mathematics is dispensable to science. But now the question arises as to why invoking the fiction of mathematics does not lead to trouble. After all, combining a scientific theory with a work of fiction would generally lead to all sorts of false and perhaps even contradictory results. What is so special about mathematics and why is it acceptable to continue using the fiction of mathematics? Field's answer is that mathematics is conservative. This means that a mathematical theory, when combined with any nominalistic scientific theory, does not yield nominalistic consequences that could not have been derived from the nominalsitic theory alone. The mathematics allows for easier derivations and the like, but enlisting it in the services of science does not yield anything new about the world. Put figuratively, the falsity of the mathematics does not infect the science that employs it. So if mathematics is conservative, we can continue using it and no damage will be done. The conservativeness claim is thus crucial in maintaining Field's contention that his fictionalism does not result in any change to scientific practice.

Why believe mathematics to be conservative? Field provides a couple of different proofs of this, but the intuitive argument in support of the claim is perhaps all we need here. Field argues that good mathematics is conservative, and a discovery that a mathematical theory was not conservative would be a reason to revise the mathematical theory in question. After all, if a mathematical theory implied statements about history or about how many biological species there are, we would look on such mathematics with extreme suspicion, even if the statements about history or biology were correct. It is also worth noting the relationship between conservativeness and some of its neighbours. Necessary truth implies truth, but it also implies conservativeness. And both the latter imply consistency. According to Field, good mathematics need not be true, but it does need to be conservative.

Various objections are levelled at Field's programme, from claims about the implausibility of extending it to non-space-time theories (e.g. quantum mechanics) (Malament 1982), through concerns about whether the nominalised science Field constructs has the theoretical virtues of its mathematical counterparts (Colyvan 2001), and whether a nominalist is entitled to be a substantivalist about space-time (Resnik 1985), to technical concerns over the logic Field relies on (first-order versus second-order logic and the account of modality) (Burgess and Rosen 1997). Despite what might seem like an overwhelming weight of criticism, it is important to recognise what Field's programme achieves. It outlines a very attractive and uncompromising fictional account of mathematics, and one that does not shirk any of the major issues. There may be problems and it may be incomplete as it stands, but Field's philosophy of mathematics is not alone in this regard.

4 Another kind of fictionalism

While many philosophers are attracted to nominalism, the difficulties facing Field's approach lead them to explore other strategies. Another approach involves rejecting the claim that we need to take all the commitments of our best scientific theories seriously. In particular, this approach denies that mathematical entities are among the entities we need to be ontologically committed to—despite their indispensability to our best scientific theories. Several contemporary philosophers have been exploring such accounts, although many are not properly regarded as nominalists (e.g. Balaguer 1998, Maddy 1997, Yablo 2005) in that they deny that there is a fact of the matter about whether mathematical entities exist. Others (e.g. Azzouni 2004) do not commit themselves to fictionalism. But still, there is a fictional account in the vicinity of the style of nominalism endorsed by the philosophers in question.

This kind of fictionalism accepts the indispensability of mathematics to science, but denies that this gives us any reason to accept the existence of mathematical entities. The reasons for this denial vary, but a common suggestion is that, in some sense, scientific theories are about physical aspects of reality, and the positing of mathematical entities is merely a tool for expressing what is required. Consider a mixed scientific statement, which invokes both mathematical entities and physical entities: "There is a continuous function that maps from the space-time manifold to the real numbers such that certain conditions are satisfied."

The fictionalist takes this statement to be false (because there are no functions) but accepts that what is said about the world is true, namely, that the space-time manifold is as described. Note that the fictionalist in question does not try to provide a mathematics-free translation of the mixed statement; that would mean commitment to something like Field's programme. There are various ways to try to motivate the non-commitment to mathematical entities. One might argue, on independent grounds, that only causally active or spatio-temporally active entities exist or, more plausibly, one might try to argue that scientific practice itself does not commit one to the existence of mathematical entities (e.g. Maddy 1997).

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See also: Abstract Objects; Fictional Entities; Fictionalism; Moral Scepticism; Realism in the Philosophy of Mathematics; Scientific Realism and Antirealism; Spacetime

References and further reading

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Eklund, M. (2007) 'Fictionalism', in E.N. Zalta (ed.) *The Stanford Encyclopedia of Philosophy*, (summer 2007 edition), URL=<http://plato.stanford.edu/archives/sum2007/entries/fictionalism/> (A useful survey of fictionalism more generally.)

Field, H. (1980) *Science Without Numbers*, Oxford: Blackwell. (A fairly technical presentation of the most influential fictionalist programme in the philosophy of mathematics.)

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