Applying Inconsistent Mathematics

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Abstract

At various times, mathematicians have been forced to work with inconsistent mathematical theories. Sometimes the inconsistency of the theory in question was apparent (e.g. the early calculus), while at other times it was not (e.g. pre-paradox naïve set theory). The way mathematicians confronted such difficulties is the subject of a great deal of interesting work in the history of mathematics but, apart from the crisis in set theory, there has been very little philosophical work on the topic of inconsistent mathematics. In this paper I will address a couple of philosophical issues arising from the applications of inconsistent mathematics. The first is the issue of whether finding applications for inconsistent mathematics commits us to the existence of inconsistent objects. I then consider what we can learn about a general philosophical account of the applicability of mathematics from successful applications of inconsistent mathematics.

1 Introduction

Inconsistent mathematics has a special place in the history of philosophy. The realisation, at the end of the 19th century, that a mathematical theory—naïve set theory—was inconsistent prompted radical changes to mathematics, pushing research in new directions and even resulted in changes to mathematical methodology. The resulting work in developing a consistent set theory was exciting and saw a departure from the existing practice of looking for self-evident axioms. Instead, following Russell [37] and Gödel [16], new axioms were assessed by their fruits.[1] Set theory shook off its foundationalist

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[1]Russell, for example, suggests that "[w]e tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true" [37, p. 273].

methodology. This episode is what philosophers live for. Philosophers played a central role in revealing the inconsistency of naïve set theory and played pivotal roles as new set theories took shape. This may have been philosophy's finest hour.[2]

Despite the importance of the crisis in set theory, inconsistent mathematics has received very little attention from either mathematicians or philosophers. Looking for inconsistency so that it might be avoided seems to be the extent of the interest. But inconsistent mathematics holds greater interest than merely providing an impetus for finding new, consistent theories to replace the old. Indeed, there are many reasons for taking inconsistent mathematical theories seriously and to be worthy of study in their own right. For example, there has been work on non-trivial, inconsistent mathematical theories such as finite models of arithmetic [20, 21, 30, 32]. While such mathematical theories might seem like mere curiosities, that's not the case. Chris Mortensen [23, 24] has argued that the best way to model inconsistent pictures (such as Penrose triangles and figures from Escher and Reutersvärd) is to invoke inconsistent geometry.[3] This work provides an interesting application for inconsistent mathematics. Although applications of inconsistent mathematics is the main theme of this paper, I want to focus on other, more mundane applications of inconsistent mathematics—applications in modelling bits of the actual world (as opposed to Escher worlds). In particular, I will argue that there are a couple of puzzles arising from the applications of inconsistent mathematics, and in both cases, the puzzles have wider implications for philosophy of mathematics.

2 Indispensability of Inconsistent Mathematical Objects

The first puzzle concerns ontology. In particular, it seems that an indispensability argument can be mounted for inconsistent (mathematical) objects. To see this, it will be useful to recall a particular, inconsistent mathematical theory, namely the early calculus.

The early calculus was inconsistent in at least two ways. First, infinitesimals were taken to be zero and non-zero. Moreover, they were taken to be zero at one place and non-zero at another place within the same proof. When dividing by infinitesimals, they were taken to be non-zero (for otherwise the

[2]See [15] for a nice account of this episode and its fallout.

[3]The consistent treatments of such figures [26, 27] do not do justice to the cognitive dissonance one experiences when viewing the figures in question *as inconsistent*.

division was illegitimate) and at other times, they were taken to be equal to zero (for example, when an infinitesimal appeared as a term in a sum). Newton, at least, tried to address such concerns by giving an interpretation of infinitesimals (or fluxions), as changing quantities. But, alas, this interpretation was itself inconsistent. After all, if an infinitesimal, δ, is a changing quantity, it cannot appear in equations such as:

$$a = a + \delta$$

where a is a constant. Why? Well, the term on the right is changing (since δ is changing) so cannot equal anything fixed, such as a constant a. Yet early calculus required equations like the one above to hold.[4]

As it turns out, the calculus could be put on a firm basis, but that didn't come until the 19th century, when Bolzano, Cauchy, and Weierstrass developed a rigourous theory of limits and the ϵ-δ notation.[5] Now the puzzle is that in the interim—over 150 years—the calculus was widely used, both in mathematics and elsewhere in science. Indeed, it is hard to imagine a more widely used and applicable theory. This presents a problem for those (like me) who take indispensability to science to be a reason to believe in the entities in question.[6]

According to this line of thought, we should be committed to the existence of all and only the entities that are indispensable to our best scientific theories and, and yet, for over 150 years, inconsistent mathematical entities—infinitesimals—were indispensable to these theories. This leads to the conclusion that we ought to have believed in the existence of inconsistent entities in the period between the late 17th century (by which time the calculus was finding widespread applications) and the middle of the 19th century

[4]In modern calculus, we'd say that the limit of $a + \delta$, as δ goes to zero, is a. But in the early days of calculus, a rigourous theory of limits was not available. Newton and Leibniz were stuck with equations like the one above. It's also worth noting that inconsistency is usually thought to be a property of formal theories and the early calculus was a long way from anything that would count as a formal theory. To claim that the early calculus was inconsistent, then, also involves some substantial claims about the interpretation of that theory *as inconsistent*. This is a big issue and much more needs to be said in order to establish beyond doubt that the early calculus was inconsistent. (For example, the early practitioners may have been groping towards one of the modern consistent interpretations of the calculus.) But it does seem that *prima facie,* at least, that both the natural interpretation and Newton's changing quantity interpretations of the early calculus were inconsistent. See [22] for more on this.

[5]Later in the 1960s work on non-standard analysis (and infinitesimals) by Robinson [36] provided a separate consistent interpretation of the calculus, and, arguable, one closer to the spirit of the original. A little later Conway [12] provided yet another way to rehabilitate infinitesimals.

[6]See [5], [8], [34] and [35] for details of the indispensability argument.

(when the calculus was finally placed on a firm foundation). It seems that if one subscribes to the indispensability argument, there's a rather unpalatable conclusion beckoning: sometimes we ought to believe in the existence of inconsistent objects [8, 9, 25].[7] Indeed, it seems that the case for inconsistent mathematical objects (in the 18th century) was every bit as good as the case for believing in consistent mathematical objects.

A couple of comments on drawing ontological conclusions from inconsistent theories. Take any inconsistent theory along with classical logic and everything is derivable, including every other contradiction and the existence of all kinds of inconsistent objects. So what do we take to be the ontological commitments of an inconsistent theory? It is clear, and well-known, that in inconsistent settings like this, a paraconsistent logic is required.[8] With such a logic in place, triviality is avoided and we can make sense of specific inconsistent objects and conclusions being entailed by the theory in question.[9] Of course, Quine would have no truck with inconsistency and paraconsistent logics, but, nevertheless, what I'm arguing for here does seem to be a very natural extension of the Quinean approach to ontology. More importantly, it is at least plausible that scientists, when working with inconsistent theories, implicitly invoke a paraconsistent logic. Of course, most working scientists (even mathematicians) don't explicitly invoke a particular logic at all. The usual story that they all use classical logic is a rational (and heavily theory-laden) reconstruction of the practice. But it is interesting to note that when contradictions arise, working mathematicians do not derive results using the familiar C.I. Lewis proof,[10] even though such proofs are classically valid. This suggests, at least, that mathematical practice might be more appropriately modelled using a paraconsistent logic, in which such proofs are invalid (disjunctive syllogism is invalid in paraconsistent logics). Of course there are other ways of explaining the practice. All I'm claiming here is that invoking paraconsistency is not as radical a move as it might first seem; it might be thought to be already implicit in mathematical practice.

On an historical note, it is interesting that both Newton and Leibniz believed that the methods of the calculus were in need of justification, and

[7] Throughout this paper I will take an inconsistent object to be an object that has inconsistent properties assigned to it by the theory positing it.

[8] This is a logic where there is some Q such that $P \wedge \neg P \nvdash Q$. That is, in paraconsistent logics not everything follows from a contradiction.

[9] And we can also deal with the related worry that there would seem to be only one inconsistent theory. As we shall see shortly, in a paraconsistent setting, we can make sense of different inconsistent theories.

[10] E.g. since an infinitesimal $\delta \neq 0$, it follows that either $\delta \neq 0$ or the fundamental theorem of calculus holds. But since $\delta = 0$, by disjunctive syllogism, we have the fundamental theorem of calculus.

both sought geometric justifications. Newton took the justification task to be that of providing a geometric proof in place of each calculus proof—calculus for discovery, but geometry for justification. Leibniz, however, took the task to be that of providing a general justification of the methods of the calculus, then business as usual [14]. Although both Newton and Leibniz were thinking in terms of justification, they can also be seen to be offering two quite different anti-realist strategies in response to the indispensability argument I just presented. Newton was advocating a kind of eliminativist strategy, whereas Leibniz was seeking a non-revisionary account. Indeed, Leibniz's quest for a general justification of the methods of the calculus has a modern-day fellow traveller in Hartry Field [13]. Leibniz sought a general geometric limit account that would ensure that the calculus, despite being inconsistent, always gave the right answers on other matters. With a bit of massaging, we can see Leibniz as seeking something like a conservativeness proof: a demonstration that the calculus was a conservative extension of standard mathematics.[11]

I have argued elsewhere [9], that it is not clear what to make of this argument for the existence of inconsistent objects. Does it tell us that consistency should be an overriding constraint in such matters? If so, why?[12] I have also (tentatively) suggested that the apparently unpalatable conclusion should be accepted: there are times when we ought to believe in inconsistent objects. But before you dismiss such thoughts as madness or perhaps as a *reductio* of the original indispensability argument, it is important to make sure that other accounts of ontological commitments do not also fall foul of inconsistent objects. Both mathematical realists and anti-realists alike have

[11]Of course, it's hard to think about conservativeness when inconsistent theories are in the mix. If we take a theory Δ to be a conservative extension of Γ, then conservativeness amounts to (roughly) that any statement formulated in the vocabulary of Γ and derivable from $\Delta + \Gamma$, is derivable from Γ alone. But if Δ is inconsistent, then it can never be conservative, so long as the logic in question is explosive (i.e. supports *ex contradictione quodlibet*). But sense can be made of conservativeness in such settings, if the logic is paraconsistent.

[12]You might think that in order for a theory to count as one of our best theories (and thus relevant to the indispensability argument), it needs to be consistent. This would rule out such cases as I'm considering here right from the start. It is hard to motivate such a privileged position for consistency, though [3, 31]. Consistency is one among many virtues theories can enjoy, but it does not seem to trump all other virtues in the way this response would require. Indeed, if I am right that scientists take inconsistent theories seriously, anyone wishing to argue that such theories are never candidates for our best theories (so no ontological conclusions can be drawn from them), would seem to be at odds with scientific practice and thereby flying in the face of philosophical naturalism. See [9] for further objections and responses to the indispensability argument I've outlined here.

always assumed the consistency of the mathematics in question. Considering inconsistent mathematical theories adds a new wrinkle to the debate over the indispensability argument, and the ontology of mathematics, more generally.

3 A Philosophical Account of Applied Mathematics

There is another, perhaps more disturbing, conclusion beckoning. If our only theories of space and time need to invoke inconsistent mathematical theories (as they did in the 18th century), this might be thought to give us reason to be realists about not just the inconsistent mathematical objects, but about the inconsistency of space and time themselves.[13] But putting such disturbing thoughts aside for the moment, let's assume that the world itself is consistent. Now there is a puzzle about how inconsistent mathematical models can be applied to the world. Again this is a new twist on an old problem.

The general problem is that of providing a philosophical account of the applicability of mathematics. This debate had its origins in the indispensability debate but has taken on a life of its own. The problem, in a nutshell, is this: how is it that mathematical structures can be so useful in modelling various aspects of the physical world.[14] The obvious answer is that when some piece of mathematics is applied to a physical system, the mathematics is applicable because there are structural similarities between the mathematical structure and the structure of the physical system. So, for example, there's no surprise that $\mathbb{R}^3$ is useful in modelling physical space, for the two are isomorphic (putting aside relativistic curvatures). But in general, isomorphism is not the appropriate structural similarity—there is usually either more structure in the world or more in the mathematics. This is where things get interesting. We need to explain how non-isomorphic structures can be used to model one another and although there are several proposals around, [1, 4, 18, 19, 28, 29] none of these is complete. The realisation that sometimes the mathematics in question is inconsistent, changes the way we might approach the problem. Assuming that the world is consistent, the problem is that of explaining how an inconsistent mathematical theory can be used to model a consistent sys-

[13]There are interesting connections here with debates about ontological vagueness (or vagueness in the world) and Russell's dismissal of it as "the fallacy of verbalism" [7]. There are also related debates about whether vagueness might give us reason to believe that the world is inconsistent [2, 10].

[14]There is also a related puzzle, often called the unreasonable effectiveness of mathematics [6, 38, 39, 40], of understanding how an apparently *a priori* discipline such as mathematics can provide the tools so often required by empirical science.

tem. This seems much tougher than explaining cases where there's simply no isomorphism, and some of the proposals do not seem well-suited to dealing with this tougher problem.

Let me make a couple of suggestions about how this problem might be solved. First, note that although the early calculus was inconsistent, it was eventually put on a firm foundation. Indeed, even when calculus was first developed, it might be argued that the consistent version existed, even though the existence of the latter wasn't known at the time. It might be further argued that this is all that's required; the inconsistent 17th century calculus is useful in applications because of its similarity to a consistent latter-day calculus.[15] The idea here is that what matters in applying mathematics is whether or not the mathematical model is capturing the salient features of the empirical phenomena in question. The model can achieve this irrespective of the knowledge of the modeller. An example might help here.

Early electrical theory had it that when there was a potential difference across a conductor, positively charged particles moved from the higher potential to the lower. This, it turns out, is wrong in a couple of ways. First, it's negatively-charged particles (electrons) that move, and in the opposite direction to that of the proposed positive particles. Second, electrons do not move very far in a conductor and they tend to oscillated—they certainly don't flow. The electrical current is the result of small movements of the electrons compounding to a net drift. So why was the original theory, which had all this wrong, so useful? It was useful because, for many purposes, these details are unimportant. The old, incorrect theory had a correct cousin—even though the latter was not known—and that's all that matters. Electrons do not care whether electricians know about them or not. There are electrons and there is (known or unknown) a correct theory of them, so all that matters is that any useful theory of electricity resembles the correct electron theory in certain respects. Clearly, positive particles flowing in one direction as opposed to negative particles flowing in the other, does not matter (unless one is specifically interested in the direction of particle movement), nor does it matter whether the particles in question flow or merely oscillate to ensure a net drift in one direction.

Returning to the case of the inconsistent calculus, we can see how the earlier suggestion might be fleshed out. It doesn't matter that the early calculus was inconsistent; it was, as a matter of fact, very similar to a consistent theory of calculus and it is this that explains the usefulness of the former. Indeed, on the account I'm proposing here, the usefulness of the calculus in

[15] Of course, nothing I've said here tells us why the latter is so useful, but the strategy here is to deal with any special issues arising from the inconsistency.

itself suggests that there is a consistent theory in the offing. And this makes good sense of several key episodes in the history of mathematics where applications helped legitimate some questionable pieces of mathematics.[16] "It works" does seem like a very good response to suspicions about a new piece of mathematics.

This seems a promising start but there are some questions to be addressed. How can a consistent theory be similar to an inconsistent one? After all, it might appear that any given consistent theory is more like an arbitrary consistent theory than any inconsistent theory. And relatedly, it might seem that there is only one inconsistent theory, since an inconsistent theory is trivial. The second worry is easily dealt with so let me tackle it first. In classical (and other explosive logics, such as intuitionistic logic), there is a sense in which there is only one inconsistent theory, namely, the trivial theory, where every proposition is true. But when dealing with inconsistency—or even potential inconsistency—we have already seen that we need to adopt a paraconsistent logic. Once this has been done, good sense can be made of *different* inconsistent theories. Seeing this also helps address the first question. Once we realise that we can discriminate between inconsistent theories, we can also determine which of these theories are similar to each other and to their consistent neighbours. It is not the case that all consistent theories are more like one another than they are to any inconsistent theory. Indeed, it is hard to see what would motivate such a thought, apart from the aforementioned mistake that there is only one inconsistent theory, namely, the trivial theory, and that this is radically unlike any consistent theory. There is still the difficult problem of how we compare theories, and nothing I've said here sheds any light on that more general problem. All I'm arguing for here is that inconsistent theories (in the context of a paraconsistent logic) can be compared in just the same way—whatever that is—to other theories, consistent and inconsistent.

The account just given seems right, in broad brush strokes, but further details will depend on the particular theory of applied mathematics adopted. So let me finish up by saying just a little about how some of the details might look in an account of the applications of mathematics I've recently developed with Otávio Bueno [4]: *the inferential conception of applied mathematics*. The full details of this theory are not important for present purposes; the basic idea is that there are three separate stages of applying mathematics. First there's the *immersion* step where a empirical set up is represented mathematically. The mathematics must be chosen in order to faithfully rep-

[16] I'm thinking here of the role of applications in helping legitimate the Dirac delta function, the early complex numbers, and, of course, the calculus [17].

resent the parts of the empirical set up that are of interest. We do not require that the mathematics is isomorphic to the empirical set up. In general, the mathematical model and the empirical set up will not be isomorphic, but some structural features will be preserved in the mathematics. The second step is the inferential step where the mathematical model is investigated and various consequences of the model are revealed. The final step is the interpretation step where the results of the inferences conducted in the mathematical model in step two are interpreted back into the empirical set up. It is important to note that the interpretation step is constrained by the immersion step—mathematically representing some physical quantity in a specific way means that one must interpret the mathematics in question as a representation of the physical quantity—but the interpretation is not just the inverse of the immersion. For instance, at the interpretation step one is free to interpret more than what was initially represented in the immersion step. It is this feature of modelling that allows the mathematics to throw up novel phenomenon for investigation. It is one of the strengths of the inferential conception of applied mathematics that it is able to make sense of this important role mathematisation plays in science.[17]

With this framework in place, we can see how inconsistent mathematical theories, such as the early calculus, might be applied. First, we might explicitly treat the world as being inconsistent in the limit. That is, we treat the world as approximately inconsistent. (This is similar to when we make other idealisations, such as treating a fluid as being approximately incompressible and model it as such, despite holding that it really is compressible.) The inconsistent mathematics is then invoked to model this inconsistent picture of the world. Indeed, the inconsistent mathematics is essential here. No consistent mathematics could model the inconsistent limit being envisaged *as an inconsistent limit*. The inferential steps, of course, would need to be conducted using a paraconsistent logic, then the results of the inferences would be interpreted as being part of the nearest consistent story (if there is one).[18] There may be more than one consistent story in the neighbourhood. If there is, all such theories would need to be considered and the question of which theory to prefer would be settled by consideration of their theoretical virtues—simplicity, unificatory power, and so on.

Alternatively, we might only discover the (implicit) inconsistent assumptions about the world after the immersion and derivation steps. So, for exam-

[17] See [4] for more details and a defence of the account.

[18] The situation here is not unlike using continuous mathematics to model discrete phenomena (e.g. differential equations in population ecology). The discrete phenomena are treated as continuous in the limit, modelled using continuous mathematics, then inferences drawn in the mathematics, and the results interpreted discretely.

ple, we might have an implicitly inconsistent theory of instantaneous change. But the inconsistencies in this theory might not be apparent until the theory is represented using calculus, and some of the consequences of the theory are revealed. Once it is realised that the theory is inconsistent, we have no reason to force the interpretation to be consistent (as we did in the case just considered). After all, in this case it was a discovery of the mathematisation process that the underlying theory is inconsistent, so it should come as no surprise that some inconsistent results are delivered. The inconsistency here is more serious than in the previous case, and may prompt further work on developing a consistent theory. In the meantime, however, we can continue using the inconsistent theory (after employing a paraconsistent logic). Again, the inconsistent mathematics is essential here. The original theory of the empirical set up was (implicitly) inconsistent—indeed, inconsistent in ways of interest (or so we are assuming here). In order to faithfully represent the theory of the empirical set up, inconsistent mathematics is needed. Any consistent mathematics used for the immersion would not only hide the inconsistencies, but would render the mathematicised theory consistent. This would make it more difficult to discover the inconsistency of the original theory. But it might seem that that's preferable. At the end of the day, we are seeking a consistent theory, so using consistent mathematics might seem like a good way to facilitate this. But this is to misunderstand the role of discovering the inconsistency. We are seeking a consistent theory, but we are not seeking *any* consistent theory. The problem with this suggestion is that using consistent mathematics to model an inconsistent theory simply renders the inconsistent theory consistent; it does not reveal the inconsistency and it does not allow for careful reflection on how best to resolve the inconsistency. It just papers over the problem. Recognising an inconsistent theory as having a specific inconsistency is an important step in securing the appropriate consistent theory.[19]

There is much more work to be done before we have an adequate philosophical account of the applications of mathematics. Although it might be tempting to ignore cases of inconsistency—both inconsistent mathematics and inconsistent empirical theories—when considering the applications of mathematics, this would be a serious mistake. As I have just shown, considering applications of inconsistent mathematics forces attention onto the nature of the structures in question and the relevant notion of similarity in a way that is enlightening—we must be able to make sense of similarity be-

[19]Something like this may have been going on with at least some of the applications of the early calculus: the underlying (unmathematised) theories of change, for example, were inconsistent in precisely the ways revealed when these theories were represented using the inconsistent calculus.

tween consistent and inconsistent structures, for example. Considering the applications of inconsistent mathematics, it seems, will help shed light on the general problem. And it is worth stressing that the inconsistent cases are not mere test cases either. A great deal of one of the most important periods in the history of science—the late 17th century to the mid 19th century—relied heavily on inconsistent mathematics. During this period, most scientists were working with an inconsistent mathematical theory and this theory was used almost everywhere. Ignoring inconsistent mathematics in a general account of applied mathematics would simply be negligence.

4 Conclusion

I have discussed just two of the many philosophical issues that arise in connection to inconsistent mathematics. Both the issues discussed in this paper revolve around applications of inconsistent mathematics. The first concerned drawing conclusions about ontological commitments from the indispensability of mathematics. When we find ourselves forced to admit the indispensability of inconsistent mathematical theories, a counterintuitive conclusion looms: sometimes we ought to believe in inconsistent objects exist. The second issue concerns the provision of an adequate account of applied mathematics—one that provides an adequate account of applications of inconsistent mathematics.

Apart from the much-discussed crisis in set theory, there has been very little work in philosophy on inconsistent mathematical theories, presumably because such mathematics is thought not to occupy a central position in mathematics itself; inconsistent mathematics is thought to be at best a curiosity or a pathological limiting case, and at worst something to be avoided at all costs. I hope this paper has gone some way to establishing that inconsistent mathematics is interesting in its own right and that including it in our stock of examples will help shed light on major issues in mainstream philosophy of mathematics.[20]

[20]I'd like to thank Otávio Bueno, Stephen Gaukroger, Øystein Linnebo, and an anonymous referee for comments on earlier drafts or for discussions that helped clarify my thinking on the issues addressed in this paper. I am also grateful to audiences at the Baroque Science Workshop at the University of Sydney in February 2008, the conference on New Waves in Philosophy of Mathematics at the University of Miami in April 2008 and at the 4th World Congress of Paraconsistency at the University of Melbourne in July 2008. Finally, I'd like to thank Chris Mortensen for impressing on me the significance of inconsistent mathematics. Work on this paper was funded by an Australian Research Council Discovery Grant (grant number DP0666020).

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