THE PURSUIT OF THE RIEMANN HYPOTHESIS

John Derbyshire, *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Washington, D.C.: Joseph Henry Press, 2003, Pp. xvi + 422. GBP19.95 HB.

By Mark Colyvan

With Fermat's Last Theorem finally disposed of by Andrew Wiles in 1994, it's only natural that popular attention should turn to arguably the most outstanding unsolved problem in mathematics: the Riemann Hypothesis. Unlike Fermat's Last Theorem, however, the Riemann Hypothesis requires quite a bit of mathematical background to even understand what it says. And of course both require a great deal of background in order to understand their significance.

The Riemann Hypothesis was first articulated by Bernhard Riemann in an address to the Berlin Academy in 1859. The address was called "On the Number of Prime Numbers Less Than a Given Quantity" and among the many interesting results and methods contained in that paper was Riemann's famous hypothesis: all non-trivial zeros of the zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, have real part 1/2. Although the zeta function as stated and considered as a real-valued function is defined only for $s > 1$, it can be suitably extended. It can, as a matter of fact, be extended to have as its domain all the complex numbers (numbers of the form $x + yi$, where x and y are real numbers and $i = \sqrt{-1}$) with the exception of $1 + 0i$ (at which point the zeta function is undefined). This extended zeta function takes the value zero for infinitely many complex numbers. For instance, all the negative even integers are zeros of the zeta function. These, however, are the *trivial* zeros. The Riemann Hypothesis is thus the conjecture that all the other zeros (and there are also infinitely many of them) have the form $1/2 + yi$. This hypothesis is of crucial importance in analytic number theory. The zeta function is very closely related to the prime counting function $\pi(N)$ (which is the number of prime numbers less than or equal to some natural number N). In fact, the zeta function "encodes" important information about the distribution of primes, and the location of the non-trivial zeros of the zeta function are crucial in all of this.

In *Prime Obsession*, John Derbyshire attempts to bring the Riemann Hypothesis within the grasp of the general popular science reader. Very little by way of prior mathematical knowledge is assumed. Derbyshire gently and patiently leads the reader through the mathematics needed. I should add that a great deal of this mathematics—basic calculus, a little complex analysis, elementary matrix algebra, and the like—will be familiar to many readers of this journal. But for the general reader, this stuff needs to be there, and (for the most part) Derbyshire pitches it at just the right level.

He resists the temptation to go deeper than necessary into the many fascinating areas of mathematics he surveys. (Though, the temptation is a little too great at a couple of points. But Derbyshire can hardly be blamed for indulging just a little in complex analysis, for instance. This is one of the most beautiful areas in all of human enquiry.) But this is not a mathematics text; it is very clearly a popular science book. And as such it skips over the more difficult mathematics. The reader is often asked to trust Derbyshire on various matters. Again this is perfectly understandable. In fact, what is impressive is how infrequently the reader is asked to do this. The patient reader will come away from this book with a very good idea of what the Riemann Hypothesis is about and why mathematicians get so excited about it. That, in my view, is a huge reward for the modest effort required.

Derbyshire also tries to give one a sense of the importance of the Riemann Hypothesis by sketching its relationship to other results in analytic number theory (results such as Prime Number Theorem: $\pi(N) \sim N/\ln N$) and even applications further afield—in quantum mechanics, for instance (see chapter 18). There is also a very nice overview of some of the recent work on solving the problem. But this is not a book just about the Riemann Hypothesis. Derbyshire also provides a great deal of historical background on some of the main characters involved in the story: Leonhard Euler, Carl Friedrich Gauss, David Hilbert, Lejeune Dirichlet, Richard Dedekind, Jacques Hadamard, J.E. Littlewood, G.H. Hardy, Bernhard Riemann (of course), and many others. The book consists of alternating chapters on the mathematics and historical background. This approach works very well. The obvious trap with such an approach is that the mathematical story and the historical story may unfold at different rates (as indeed they do) and that this will lead to continuity problems. The latter is not the case here though. Derbyshire has obviously invested a great deal of effort into getting the alternating chapter structure to hang together and, in particular, into finding appropriate segues from one chapter to the next. This, I should add, was done without compromising the natural pace of the various stories.

Less successful, I thought were some other concessions to the popular science genre. For instance, while the example of a 12-hour clock to illustrate modular arithmetic is helpful, the continued reference to modular arithmetic as "clock arithmetic" I found annoying. Any reader interested enough to read a book like this and patient enough to work through the mathematics presented I think is capable of understanding modular arithmetic (and have it referred to by its rightful name). Also Derbyshire's rather silly clock notation for rings (page 268) I found distracting. And while on the topic of minor annoyances, I found the repeated reference to the Euler Product Formula, $\sum_n n^{-s} = \prod_p (1-p^{-s})^{-1}$, as "The Golden Key" a little sensationalist and unnecessary. Still, as I've pointed out, the book is intended for a popular audience, and perhaps some readers appreciate such attempts at making the mathematics friendlier.

But these minor complaints aside, I enjoyed reading *Prime Obsession*. Although the book is popular in style, it does a very good job of laying out most of the technical background required to understand one of the most important (if not *the* most important) open questions in mathematics. Derbyshire is to be congratulated for making such important mathematics accessible to non-specialists.

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