TRUTH IN MATHEMATICS

H.G. Dales and G. Oliveri (eds.)

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Reviewed by Mark Colyvan

The question of truth in mathematics has puzzled mathematicians and philosophers for centuries. On the one hand, it is hard to take seriously doubts about the truth of (at least some of) our mathematical beliefs. No-one who understands the language in question could doubt that there exists an even prime number, and no-one who understands Pythagoras's famous proof could doubt that $\sqrt{2}$ is irrational. Such beliefs, it seems, constitute some of our best candidates for knowledge. On the other hand, though, it's hard to understand what makes mathematical sentences true. We can provide a plausible account of truth for sentences such as 'Hobart is in Tasmania'—there is a city called 'Hobart' and it has the property of being in Tasmania. (This is essentially the approach developed by Tarski [11].) A similar account of mathematical truth, however, seems to run into problems.

Let's suppose, for example, that the sentence 'seven is prime' is made true by the existence of the number seven and it having the property of primeness. (Call this *the Platonist response*.) There are various well-known problems with this position. Do we really believe that numbers exist (in the same sense that we believe that Hobart exists)? If we do, then we have the problem of explaining how we come by knowledge of numbers (which are, on this account, taken to be mind independent, abstract objects) by apparently *a priori* means. The acausal nature of mathematical entities is the heart of the problem here. The fact that they do not cause *anything*, means that they cannot impact on our sense organs and so it seems utterly mysterious how we come by mathematical knowledge. This is the so-called epistemological problem for Platonism, presented in Paul Benacerraf's 1973 paper [3]. So serious is this problem that despite the intuitive appeal of the Platonist response to mathematical truth, some have rejected Platonism in favour of various anti-Platonist positions such as *formalism* (mathematics is nothing more than a formal game of symbol manipulation) and *fictionalism* (mathematical statements such as 'seven is prime' are, strictly speaking, false, but *true in the story of mathematics*).

In September 1995 mathematicians and philosophers from around the world converged on the beautiful hilltop town of Mussomeli in central Sicily for a conference devoted to the problems associated with truth in mathematics. The present volume, edited by Garth Dales and Gianluigi Oliveri, is the proceedings of that conference. It contains 18 new essays (including a very useful introduction by Dales and Oliveri). These essays fall under one of four section headings: 1. Knowability, Constructivity, and Truth; 2. Formalism and Naturalism; 3. Realism in Mathematics; and 4. Sets, Undecidability, and the Natural Numbers. The book contains essays both by leading philosophers of mathematics and leading mathematicians. Indeed, one of the strengths of the conference, and this is reflected in this volume, was the genuine spirit of collaboration between the philosophical and mathematical communities in addressing the important issues associated with truth in mathematics.

Of course I can't do justice to all the fascinating papers contained in this volume in a review such as this. Instead I'll discuss some of the papers that touched on one of the recurring themes of the conference: the independent questions of set-theory. These are questions that have more than one answer consistent with the standard ZFC axioms. Examples of such questions include the Lebesgue measurability of Σ^1_2 sets in the projective hierarchy and Cantor's continuum problem—Does $2^{\aleph_0} = \aleph_1$? (and its generalisation—Does $2^{\aleph_\alpha} = \aleph_{\alpha+1}$?). Let's consider the continuum hypothesis: $2^{\aleph_0} = \aleph_1$. It is well known that theorems due to Gödel and Cohen demonstrate that both the continuum hypothesis and its negation are consistent with ZFC (if ZFC is consistent). What does this tell us about the truth of the continuum hypothesis? One answer is that there is nothing more to set theory than the ZFC axioms, so the continuum hypothesis is neither true nor false—it is genuinely indeterminate. (The continuum problem might even be taken to be, in some sense, *illegitimate*.) Another answer is that the independence of the continuum hypothesis from ZFC suggests that ZFC is not the complete description of the set-theoretic universe. Various new axiom candidates have been put forward in this regard, including Gödel's axiom of constructibility—V=L—and a variety of large cardinal axioms such as MC (there exists a measurable cardinal). Moreover, the new axiom candidates, when added to ZFC, give different answers to the continuum question. (For example, with ZFC + V=L, $2^{\aleph_0}$ does indeed equal $\aleph_1$ but with ZFC + MC, $2^{\aleph_0} \neq \aleph_1$.) The interesting issue here is what counts as evidence for the axiom candidates in question.

Penelope Maddy, in her paper in this volume, 'How to be a Naturalist about Mathematics', looks at the debate over the open questions and how these debates relate to questions about the existence of sets. In particular, she is interested to give a philosophical account of set theory that pays due respect to mathematical methodology. That is, the account must respect the standards of evidence accepted by working mathematicians when approaching the question of what the correct/best new axiom is. For instance, the first response to the independent questions that I mentioned above might suggest that *any* extension of ZFC is no more or no less correct than any other: "Let a hundred flowers bloom". Accept ZFC + V=L *and* ZFC + MC. Just as when Hollywood movie directors film two endings, they are not interested in which is the "correct" ending (whatever that means), they are merely interested in which will win at the box office. But this approach seems to make truth irrelevant to mathematics. Furthermore, it is hard to reconcile such an account with mathematical practice; most working set theorists think that there is a real issue here.

Maddy argues for what she calls 'Set-theoretic Naturalism'. This approach, she claims, pays respect to standard mathematical methodology without committing to dubious metaphysical assumptions such as the existence of sets (or the non-existence of sets, for that matter). (The interested reader should also see Maddy's recent book [5] in which she has a lengthy discussion of the mathematical debates over various axioms including Choice, and V=L.)

Closely related to the debates over the independent questions of ZFC, is the matter of what counts as evidence for mathematical propositions. Donald Martin takes up this latter question in his paper in this volume, 'Mathematical Evidence'. The first and most obvious source of evidence is proof. Martin points out that we can state rather precisely what a mathematical proof of some proposition S is:

> To prove S, one must show that S follows by pure logic from the basic principles of mathematics. It is one of the triumphs of modern logic that one can say precisely what 'pure logic' is, in the relevant sense: namely, first-order logic. And it is a rather surprising fact that one can say precisely what the 'basic principles of mathematics' are: namely, the Zermelo–Frænkel (ZFC) axioms for set theory. (p. 216)

Modulo some concerns about alternate logics and alternate set theories, Martin is right about proof. But this just serves to highlight the point that each of the ZFC axioms are unproven (except in the trivial sense that each can be proven from itself). Martin is primarily concerned with the question of what counts as evidence for them. As a case study, he investigates the evidence for an important class of new axiom candidates called *determinacy hypotheses*. (For those familiar with the area, Martin has in mind Projective Determinacy and $\mathrm{AD}^{L(\mathbb{R})}$.) Since none of these is generally accepted as a basic axiom (yet), the alleged evidence for determinacy hypotheses sheds some light on what should and what should not count as evidence for new axioms. Martin's conclusion, that we should look beyond self-evidence and naturalness, echoes a famous passage of Gödel:

> There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. [4, p. 477]

Hartry Field is another who takes up the discussion of undecidable questions in his article 'Which Undecidable Mathematical Sentences have Determinate Truth Values?'. Field's concerns are a little closer to home than the higher reaches of set theory though. He is concerned with the truth values of various undecidable statements of elementary number theory. Such statements, claims Field, strike us as having determinate truth values though we know not what they are. The reason for this is that we feel that we have a determinate grasp of the concept of 'only finitely many'. Field shows how we can hold onto determinate truth values for all of number theory, despite Gödel's incompleteness theorem, so long as we can show that the quantifier 'only finitely many' is determinate. He then goes on to suggest a way in which this might be done by appeal to certain cosmological assumptions about time. This proposal may seem quite bizarre, and Field himself acknowledges this:

> It might be thought objectionable to use physical hypotheses to secure the determinacy of mathematical concepts like finiteness. I sympathize—I just do not know of any other way to secure their determinacy. (p. 299)

The indeterminacy problem that concerns Field is that of ruling out nonstandard models of arithmetic. If we cannot do this, we can hardly claim to have a grasp of the extension of ordinary mathematical functions such as 'successor'. (Field takes the most compelling form of this indeterminacy argument to be due to Hilary Putnam [6].) This is not the place to discuss these issues in detail but suffice to say that Field presents a novel solution to one of the most disturbing indeterminacy arguments in the philosophy of mathematics.

Although, for the most part, I have been discussing the independent questions of set theory and their consequences for mathematical truth, I do not wish to give the impression that all the essays in this volume are devoted to this topic. (Some of the other topics discussed are intuitionistic and constructivist mathematics, formalism, consistency, realism, and rigour.) Indeed, even those primarily concerned with the independent questions of set theory touch on many other related areas, both philosophical and mathematical.

Let me finish with a brief discussion of one other topic addressed in this volume: the question of the appropriate standards of rigour in mathematical proof. Vaughan Jones is one who takes up this topic, in his paper 'A Credo of Sorts'. He discusses how standards of proof vary in different branches of mathematics and he argues that proofs are necessary for belief in the truth of mathematical theorems but they are not sufficient. He illustrates this with an example of debugging a recalcitrant computer program:

> To write a short program, say 100 lines of C code, is a relatively painless experience. The debugging will take longer than the writing, but it will not involve suicidal thoughts. However, should an inexperienced programmer undertake to write a slightly longer program, say 1000 lines, distressing results will follow. The debugging process becomes an emotional nightmare in which one will doubt one's own sanity. One will certainly insult the compiler in words that are inappropriate for this essay. The mathematician, having gone through this torture, cannot but ask: "Have I ever subjected the proofs of any of my theorems to such close scrutiny?" In my case at least the answer is surely "no". So while I do not doubt that my proofs are correct (at least the significant ones), my belief in the results needs bolstering. Compare this with the debugging process. At the end of debugging we are happy with our program because of the consistency of the output it gives, *not* because we feel we have proved it correct—after all we did that at least twenty times while debugging and we were wrong *every time*. (p. 208)

It should be clear by now that I think this is a very interesting and important collection of essays, and is a valuable addition to the growing, contemporary literature on the philosophy of mathematics. (See, for example, [1], [2], [7], [8], [9] and [10].) It will be of interest to mathematically-minded philosophers for both the contributions from leading philosophers (such as Hartry Field) but also for the insights gained from the essays by leading mathematicians (such as Vaughan Jones). Despite being unashamedly philosophical (even those articles by mathematicians are philosophical), I think that many mathematicians will also enjoy this volume. If for no other reason, it helps to clarify one of the most basic questions one can ask about mathematics: What is mathematical truth and how do we recognise it? Just as importantly, the book brings together two quite different cultures—mathematics and philosophy—and reinforces the need for collaboration in solving problems that are fundamental to both disciplines.

References

[1] Balaguer, M. *Platonism and Anti-Platonism in Mathematics*, Oxford University Press, New York, 1998.

[2] Burgess, J.P. and Rosen, G. *A Subject with No Object: Strategies for Nominalistic Interpretation of Mathematics*, Clarendon, New York, 1997.

[3] Benacerraf, P. 'Mathematical Truth', reprinted in Benacerraf, P. and Putnam, H. (eds.) *Philosophy of Mathematics: Selected Readings*. Second edition. Cambridge University Press, Cambridge, 1983, 403–420.

[4] Gödel, K. 'What is Cantor's Continuum Problem?', reprinted and expanded in Benacerraf, P. and Putnam, H. (eds.) *Philosophy of Mathematics: Selected Readings*. Second edition. Cambridge University Press, Cambridge, 1983, 470–85.

[5] Maddy, P. *Naturalism in Mathematics*, Clarendon, Oxford, 1997.

[6] Putnam, H. 'Models and Reality', *Journal of Symbolic Logic*, Vol. 45 (1980), 464–82.

[7] Resnik, M. *Mathematics as a Science of Patterns*, Clarendon, Oxford, 1997.

[8] Schirn, M. (ed.) *The Philosophy of Mathematics Today*, Clarendon, Oxford, 1998.

[9] Shapiro, S. *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press, New York, 1997.

[10] Steiner, M. *The Applicability of Mathematics as a Philosophical Problem*, Harvard University Press, Cambridge, MA, 1998.

[11] Tarski, A. 'The Semantic Conception of Truth and the Foundations of Semantics', reprinted in A.P. Martinich, *The Philosophy of Language*, 3rd edition, Oxford University Press, Oxford, 1996.

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